Universal Central Extension of the Lie Algebra of Hamiltonian Vector Fields

Bas Janssens[*], Cornelia Vizman[†]

December 21, 2016

Abstract

For a connected symplectic manifold X, we determine the universal central extension of the Lie algebra $\mathrm{ham}(X)$ of hamiltonian vector fields. We classify the central extensions of $\mathrm{ham}(X)$, of the Lie algebra $\mathrm{sp}(X)$ of symplectic vector fields, of the Poisson Lie algebra $C^\infty(X)$, and of its compactly supported version $C^\infty_c(X)$.

1 Introduction

In this paper, we classify continuous central extensions of several infinite dimensional Lie algebras associated to a symplectic manifold (X, ω); the Poisson Lie algebra $C^\infty(X)$, the compactly supported Poisson Lie algebra $C^\infty_c(X)$, the Lie algebra of hamiltonian vector fields $\mathrm{ham}(X)$, and the Lie algebra $\mathrm{sp}(X)$ of symplectic vector fields.

If $\mathfrak{g}$ is the Lie algebra of a locally convex Lie group G, then central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are the infinitesimal versions of central extensions of G by $U(1)$, whose classification plays a pivotal role in projective unitary representation theory [PS86, Nee02a, JN15]. If X is compact, then the Lie algebra of hamiltonian or symplectic vector fields is the Lie algebra of the group of hamiltonian or symplectic diffeomorphisms, respectively. If, moreover, ω is integral, then the Poisson Lie algebra is the Lie algebra of the quantomorphism group [Sch78, RS81]. Although the link with Lie groups is important for applications, the present paper is only concerned with Lie algebras, so we will be able to handle noncompact as well as compact manifolds X. The integrability issue is addressed in a forthcoming paper.

Continuous central extensions of a locally convex Lie algebra $\mathfrak{g}$ by $\mathbb{R}$ are classified by the continuous second Lie algebra cohomology $H^2(\mathfrak{g}, \mathbb{R})$ with coefficients in $\mathbb{R}$. Therefore, a large part of this paper is devoted to determining this cohomology group for the above mentioned Lie algebras associated to a symplectic manifold X.

This problem is somewhat different in flavour from determining the cohomology with adjoint coefficients, cf. [Lic73, Lic74, DWL83b, DWL83a, DWLG84],

[*]Mathematisch Instituut, Universiteit Utrecht, 3584 CD Utrecht, the Netherlands. Email: B.Janssens@uu.nl

[†]Department of Mathematics, West University of Timişoara, 300223-Timişoara Romania. Email: cornelia.vizman@e-uvt.ro

in the sense that our 2-cocycles come from distributions on $X \times X$ that can be singular in all directions. Cohomology with real coefficients was studied in [GKF72, GM92] for $\mathrm{ham}(X)$, where X is a formal neighbourhood or a torus, respectively. Lie algebra cohomology for various Lie algebras of vector fields is treated in [Fuk86]. Our description of the second cohomology goes back to Kirillov [Kir90] (for dimension 2) and Roger [Rog95] (for arbitrary dimension), but no proof has appeared in the literature to the best of our knowledge.

1.1 Summary of the Results

We give a brief summary of our main results: the continuous second Lie algebra cohomology of $C^\infty_c(X)$, $C^\infty(X)$, $\mathrm{ham}(X)$, and $\mathrm{sp}(X)$, and the universal central extension of $\mathrm{ham}(X)$ for a symplectic manifold (X, ω). It turns out that for X noncompact, the same Lie algebra serves as the universal central extension of the Poisson Lie algebra $C^\infty(X)$.

1.1.1 Continuous second Lie algebra cohomology

The second Lie algebra cohomology of $C^\infty_c(X)$, $C^\infty(X)$, and $\mathrm{ham}(X)$ is determined in Section 4. The case $\mathrm{sp}(X)$ is based on this, but requires also some different methods. It is done in Section 6.

Compactly supported Poisson Lie algebra. Let $C^\infty_c(X)$ be the Lie algebra of compactly supported, smooth functions on a symplectic manifold, equipped with the Poisson bracket. Then Theorem 4.15 shows that the central extensions of $C^\infty_c(X)$ by $\mathbb{R}$ are classified by the first de Rham cohomology of X,

$$H^2(C^\infty_c(X), \mathbb{R}) \simeq H^1_{\mathrm{dR}}(X)\,. \tag{1}$$

This theorem underlies all other results in this paper.

Poisson Lie algebra. Let $C^\infty(X)$ be the Lie algebra of smooth functions on X, equipped with the Poisson bracket. Then Theorem 4.16 shows that the central extensions of $C^\infty(X)$ by $\mathbb{R}$ are classified by the compactly supported first de Rham cohomology of X,

$$H^2(C^\infty(X), \mathbb{R}) \simeq H^1_{\mathrm{dR},c}(X)\,. \tag{2}$$

Hamiltonian vector fields. Let $\mathrm{ham}(X)$ be the Lie algebra of hamiltonian vector fields. Then Theorem 4.17 shows that continuous central extensions of $\mathrm{ham}(X)$ by $\mathbb{R}$ are classified by

$$H^2(\mathrm{ham}(X), \mathbb{R}) \simeq Z^1_c(X)/d\Omega^0_{c,0}(X)\,, \tag{3}$$

where $Z^1_c(X)$ is the space of closed, compactly supported 1-forms on X, and $\Omega^0_{c,0}(X)$ is the space of compactly supported smooth functions f on X that integrate to zero, $\int_X f\omega^n/n! = 0$. If X is compact, then (3) reduces to

$$H^2(\mathrm{ham}(X), \mathbb{R}) \simeq H^1_{\mathrm{dR}}(X)\,, \tag{4}$$

as one can always change f by a suitable constant to make its integral zero. If X is not compact, then there is precisely one 'extra' central extension. The

1-dimensional kernel of the canonical surjection $Z^1_c(X)/d\Omega^0_{c,0}(X) \to H^1_{\mathrm{dR},c}(X)$ is spanned by the Kostant-Souriau class $[\psi_{KS}]$, so that for noncompact X, the expression (3) reduces to

$$H^2(\mathrm{ham}(X), \mathbb{R}) \simeq H^1_{\mathrm{dR},c}(X) \oplus \mathbb{R}[\psi_{KS}]\,. \tag{5}$$

The class $[\psi_{KS}]$ corresponds to the central extension $\mathbb{R} \to C^\infty(X) \to \mathrm{ham}(X)$, which is trivial for compact X.

The result (4) for compact X was stated in [Rog95], but we are not aware of any proof in the literature so far.

Symplectic vector fields. To describe the continuous second Lie algebra cohomology of the Lie algebra $\mathrm{sp}(X)$ of symplectic vector fields, introduce the bilinear form $H^1_{\mathrm{dR},c}(X) \times H^1_{\mathrm{dR}}(X) \to \mathbb{R}$, defined by

$$([\alpha], [\beta]) := \int_X \alpha \wedge \beta \wedge \omega^{n-1}/(n-1)!,$$

and the 4-linear form $H^1_{\mathrm{dR},c}(X) \times \wedge^3 H^1_{\mathrm{dR}}(X) \to \mathbb{R}$, defined by

$$([\alpha], [\beta_1], [\beta_2], [\beta_3]) := \int_X \alpha \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \omega^{n-2}/(n-2)!.$$

If X is compact and connected, Theorem 6.7 shows that

$$H^2(\mathrm{sp}(X), \mathbb{R}) \simeq \textstyle\bigwedge^2 H^1_{\mathrm{dR}}(X)^* \oplus \mathrm{K_c}, \tag{6}$$

where $K_\mathrm{c} \subseteq H^1_{\mathrm{dR}}(X)$ is the set of classes $a \in H^1_{\mathrm{dR}}(X)$ such that

$$(a, b_1, b_2, b_3) = \frac{1}{\mathrm{vol}(X)} \sum_{\mathrm{cycl}} (a, b_1)(b_2, b_3)$$

for all $b_1, b_2, b_3 \in H^1_{\mathrm{dR}}(X)$.

If X is noncompact and connected, Theorem 6.13 shows that

$$H^2(\mathrm{sp}(X), \mathbb{R}) \simeq \textstyle\bigwedge^2 H^1_{\mathrm{dR}}(X)^* \oplus \mathbb{R}[\psi'_{KS}] \oplus \mathrm{K_{nc}}\,, \tag{7}$$

where $\psi'_{KS}(v, w) = \omega(v, w)_x$ is an extension to $\mathrm{sp}(X)$ of the Kostant-Souriau cocycle ψ_{KS}, and $\mathrm{K_{nc}} \subseteq H^1_{\mathrm{dR},c}(X)$ is the set of classes a such that $(a, b) = 0$ and $(a, b_1, b_2, b_3) = 0$ for all b and b_1, b_2, b_3 in $H^1_{\mathrm{dR}}(X)$.

The case of compact, connected X was treated in [Viz06] under the assumption that $H^2(\mathrm{ham}(X), \mathbb{R}) \simeq H^1_{\mathrm{dR}}(X)$, which is justified by the present paper. The case of noncompact, connected X appears to be new.

1.1.2 The universal central extension

A Lie algebra possesses a universal central extension if and only if it is perfect [vdK73]. Using results in [ALDM74], we show in Corollary 3.2, Proposition 3.3 and Corollary 3.7 that of the above Lie algebras, only $\mathrm{ham}(X)$ and $C^\infty(X)$ are perfect; the former for any symplectic manifold, the latter only if X has no compact connected components.

$\mathfrak{g}$	X	$H^2(\mathfrak{g},\mathbb{R})$
$C_c^\infty(X)$	compact	$H^1_{\mathrm{dR}}(X)$
	noncompact	$H^1_{\mathrm{dR}}(X)$
$C^\infty(X)$	compact	$H^1_{\mathrm{dR}}(X)$
	noncompact	$H^1_{\mathrm{dR,c}}(X)$
$\mathrm{ham}(X)$	compact	$H^1_{\mathrm{dR}}(X)$
	noncompact	$H^1_{\mathrm{dR,c}}(X)\oplus\mathbb{R}$
$\mathrm{sp}(X)$	compact	$\Lambda^2 H^1_{\mathrm{dR}}(X)^*\oplus K_c$
	noncompact	$\Lambda^2 H^1_{\mathrm{dR}}(X)^*\oplus K_{nc}\oplus\mathbb{R}$

Table 1: Results on second continuous Lie algebra cohomology

We describe the universal central extension of these two Lie algebras. Let $*:\Omega^k(X)\to\Omega^{2n-k}(X)$ be the *symplectic Hodge star operator*, which is defined like the ordinary Hodge star operator, with the Riemannian form replaced by the symplectic form. Its defining property is that for all $\alpha,\beta\in\Omega^k(X)$, $\beta\wedge*\alpha=(\wedge^k\omega)(\beta,\alpha)\omega^n/n!$, where $\wedge^k\omega$ is the $(-1)^k$-symmetric form induced by ω on $\wedge^k T^*X$.

From the de Rham differential d, one then obtains the *canonical homology operator* $\delta := (-1)^{k+1} * d\,*$, which lowers rather than raises the degree, $\delta\colon\Omega^k(X)\to\Omega^{k-1}(X)$. The universal central extension is the Fréchet Lie algebra

$$\Omega^1(X)/\delta\Omega^2(X)\,, \tag{8}$$

with Lie bracket

$$[[\alpha],[\beta]] := [\delta\alpha\cdot d\delta\beta]\,. \tag{9}$$

This is the universal central extension of $\mathrm{ham}(X)$ for any connected, symplectic manifold X, and also of $C^\infty(X)$ if X is noncompact.

Universal central extension for hamiltonian vector fields. To obtain the universal central extension of $\mathrm{ham}(X)$, equip $\Omega^1(X)/\delta\Omega^2(X)$ with the projection

$$\mathfrak{z}\longrightarrow\Omega^1(X)/\delta\Omega^2(X)\xrightarrow{\ \pi\ }\mathrm{ham}(X)\,, \tag{10}$$

defined by $\pi([\alpha]) := X_{\delta\alpha}$, where X_f is the hamiltonian vector field corresponding to $f\in C^\infty(X)$. The kernel of π is precisely the centre $\mathfrak{z}=\mathrm{Ker}(d\circ\delta)/\delta\Omega^2(X)$ of $\Omega^1(X)/\delta\Omega^2(X)$. If X is compact, then $\mathfrak{z}$ is the first canonical homology

$$H_1^{\mathrm{can}}(X) := \mathrm{Ker}(\delta\colon\Omega^1(X)\to\Omega^0(X))/\mathrm{Im}(\delta\colon\Omega^2(X)\to\Omega^1(X))\,,$$

which is isomorphic to the de Rham cohomology $H^{2n-1}_{\mathrm{dR}}(X)$, with $\dim(X)=2n$, by the Hodge star operator. If X is noncompact, then $H^{2n-1}_{\mathrm{dR}}(X)\subset\mathfrak{z}$ is a subspace of codimension 1.

Theorem 5.6 shows that for X connected, the central extension (10) is universal for continuous extensions of $\mathrm{ham}(X)$ by a finite dimensional centre, and, if $H^{2n-1}_{\mathrm{dR}}(X)$ is finitely generated, even for linearly split continuous extensions by an infinite dimensional centre. The requirement that a Lie algebra extension

be linearly split is quite natural, and automatically satisfied if it comes from a Lie group extension.

Universal central extension for Poisson Lie algebra. For a noncompact, connected, symplectic manifold X, the universal central extension of $C^{\infty}(X)$ is again $\Omega^1(X)/\delta\Omega^2(X)$, but with a different projection. Corollary 5.8 shows that the central extension

$$H_1^{\mathrm{can}}(X) \to \Omega^1(X)/\delta\Omega^2(X) \xrightarrow{\delta} C^{\infty}(X) \tag{11}$$

is universal for continuous extensions of the Poisson Lie algebra $C^{\infty}(X)$ by a finite dimensional centre, and, if $H_1^{\mathrm{can}}(X)$ is finite dimensional, also for linearly split continuous extensions by an infinite dimensional centre.

Roger cocycles and singular cocycles. We describe two particularly relevant classes of 2-cocycles on $\mathrm{ham}(X)$, related to de Rham cohomology and singular homology of X.

By the universal property of (10) for a connected, symplectic manifold X, continuous 2-cocycles ψ of $\mathrm{ham}(X)$ correspond bijectively to continuous linear functionals S on $\Omega^1(X)/\delta\Omega^2(X)$ by

$$\psi_S(X_f, X_g) = S([fdg])\,. \tag{12}$$

In order to describe the isomorphism (3), it is most convenient to consider the class of *Roger cocycles* ψ_{α}, related to de Rham cohomology of X [Rog95]. They come from a compactly supported, closed 1-form $\alpha \in \Omega^1_c(X)$. Via the functional $S_{\alpha}([\beta]) := \int_X *\beta \wedge \alpha$, they yield the 2-cocycle

$$\psi_{\alpha}(X_f, X_g) = S_{\alpha}([fdg]) = \int_X f(i_{X_g}\alpha)\omega^n/n!\,. \tag{13}$$

The isomorphism $Z^1_c(X)/d\Omega^0_{c,0}(X) \simeq H^2(\mathrm{ham}(X), \mathbb{R})$ is then given by $[\alpha] \mapsto [\psi_{\alpha}]$.

We also introduce a class of *singular cocycles* ψ_N that are expected to be of particular relevance in the projective unitary representation theory of $\mathrm{ham}(X)$. They come from $(2n-1)$-dimensional submanifolds N of X. The functional $S_N([\beta]) = \int_N *\beta$ on $\Omega^1(X)/\delta\Omega^2(X)$ yields the 2-cocycle

$$\psi_N(X_f, X_g) = S_N([fdg]) = \int_N fdg \wedge \omega^{n-1}/(n-1)!\,. \tag{14}$$

1.2 Outline

Having already summarised the main results of the present paper, we can be brief about the outline.

In Section 2, we fix our notation and introduce the Hodge star operator. We describe the relevant notions of central extensions and Lie algebra cohomology, and prove some simple but useful lemmas.

In Section 3, we determine the characters of the Lie algebras associated to X, which amounts to determining their first cohomology with trivial coefficients. We prove that $\mathrm{ham}(X)$ is perfect, as is $C^{\infty}(X)$ for X noncompact.

Section 4 is the heart of the paper. We prove that all continuous 2-cocycles are diagonal, and hence described by differential operators. We then show that they are of order at most 1. We investigate how the cocycles of the different Lie algebras are related to each other. Starting from $C_c^\infty(X)$, we derive the specific form of the cocycles for the Lie algebras associated to X.

In Section 5, we give a detailed description of the universal central extension $\Omega^1(X)/\delta\Omega^2(X)$ of $\mathrm{ham}(X)$, and show that it also serves as a universal central extension of $C^\infty(X)$ if X is noncompact.

In Section 6, we derive $H^2(\mathrm{sp}(X),\mathbb{R})$ from the results in Section 4 using two similar, but slightly different transgression arguments, one for compact and one for noncompact manifolds X.

2 Preliminaries

In this section, we collect some useful and essentially well known results, and adapt them to our setting. In Section 2.1, we fix some notation. In Section 2.3, we give the precise link between (universal) central extensions of locally convex Lie algebras and continuous Lie algebra cohomology. In Section 2.2, we describe the Hodge star operator and canonical cohomology of symplectic manifolds. Finally, in Section 2.4, we collect some basic lemmas that will be used later on.

2.1 Notation and Conventions

We fix some notation and conventions that will be used throughout the paper. Let (X,ω) be a (second countable) symplectic manifold of dimension $2n$, not necessarily connected unless specified otherwise. Then the Lie algebra of *symplectic vector fields* is denoted by

$$\mathrm{sp}(X) := \{v \in \mathrm{vec}(X)\,;\, L_v\omega = 0\}\,.$$

The *hamiltonian vector field* X_f of $f \in C^\infty(X)$ is the unique vector field such that $df = -i_{X_f}\omega$. Since the Lie bracket of $v, w \in \mathrm{sp}(X)$ is hamiltonian with $-i_{[v,w]}\omega = d\omega(v,w)$, the Lie algebra of hamiltonian vector fields

$$\mathrm{ham}(X) := \{X_f\,;\, f \in C^\infty(X)\}$$

is an ideal in $\mathrm{sp}(X)$. The map $X \mapsto -[i_X\omega]$ identifies the quotient of $\mathrm{sp}(X)$ by $\mathrm{ham}(X)$ with the abelian Lie algebra $H^1_{\mathrm{dR}}(X)$,

$$0 \to \mathrm{ham}(X) \to \mathrm{sp}(X) \to H^1_{\mathrm{dR}}(X) \to 0\,. \tag{15}$$

We equip the space $C^\infty(X)$ of $\mathbb{R}$-valued, smooth functions on X with the *Poisson bracket* $\{f,g\} = \omega(X_f, X_g)$. The kernel of the Lie algebra homomorphism $f \mapsto X_f$, the space $H^0_{\mathrm{dR}}(X)$ of locally constant functions, is central in $C^\infty(X)$. We thus we obtain a central extension

$$0 \to H^0_{\mathrm{dR}}(X) \to C^\infty(X) \to \mathrm{ham}(X) \to 0\,. \tag{16}$$

We also consider the subalgebra $C^\infty_c(X)$ of *compactly supported* functions. The map $C^\infty_c(X) \to H^{2n}_{\mathrm{dR},c}(X)$ defined by $f \mapsto [f\omega^n/n!]$ is a Lie algebra homomorphism into an abelian Lie algebra; every commutator $\{f,g\} = i_{X_f}dg$ maps

to zero, as $(i_{X_f}dg)\omega^n/n! = dg \wedge (i_{X_f}\omega^n/n!) = d(gi_{X_f}\omega^n/n!)$ is exact. We define the ideal $C^\infty_{c,0}(X)$ to be the kernel of this map,

$$C^\infty_{c,0}(X) := \{f \in C^\infty_c(X)\,;\, 0 = [f\omega^n/n!] \in H^{2n}_{\mathrm{dR},c}(X)\}\,, \tag{17}$$

so that we obtain an exact sequence

$$0 \to C^\infty_{c,0}(X) \to C^\infty_c(X) \to H^{2n}_{\mathrm{dR},c}(X) \to 0\,. \tag{18}$$

If X is connected, $C^\infty_{c,0}(X)$ is the space of zero-integral functions, $\int_X f\omega^n/n! = 0$.

2.2 The symplectic hodge star operator

The *symplectic Hodge star operator* $* : \Omega^k(X) \to \Omega^{2n-k}(X)$ is uniquely defined by the requirement that $\beta \wedge *\alpha = (\wedge^k\omega)(\beta,\alpha)\omega^n/n!$ for all $\beta \in \Omega^k(X)$, where $\wedge^k\omega$ is the $(-1)^k$-symmetric form induced by ω on $\wedge^k T^*X$. Alternatively, it is described by contraction with the tensor

$$T = \omega^{\mu_1\nu_1}\cdots\omega^{\mu_k\nu_k}\,\partial_{\mu_1}\wedge\cdots\wedge\partial_{\mu_k}\otimes i_{\partial_{\nu_k}}\cdots i_{\partial_{\nu_1}}\omega^n/n!.$$

We define the *canonical homology operator* $\delta\colon \Omega^k(X) \to \Omega^{k-1}(X)$ by

$$\delta := (-1)^{k+1} * d\, *\ . \tag{19}$$

In the following theorem, we gather some facts about $*$ and δ that we will need later.

Theorem 2.1 (Brylinski)**.** *The symplectic star operator satisfies the equalities*

$$\begin{aligned} * f_0 df_1 \wedge \ldots \wedge df_k &= (-1)^k f_0 i_{X_{f_k}} \ldots i_{X_{f_1}}\omega^n/n! \quad \text{and} && (20)\\ *^2 &= \mathrm{Id}. && (21)\end{aligned}$$

Furthermore, if $\pi \in \Gamma(\wedge^2 TX)$ is the Poisson bivector field defined by ω, then $\delta = i_\pi \circ d - d \circ i_\pi$. For all $f_0,\ldots,f_k \in C^\infty(X)$, we thus have

$$\begin{aligned}\delta(f_0df_1\wedge\ldots\wedge df_k) &= \sum_{i=1}^k(-1)^{i+1}\{f_0,f_i\}df_1\wedge\ldots d\hat{f_i}\ldots\wedge df_k && (22)\\ &+ \sum_{1\le i<j\le k}(-1)^{i+j}f_0d\{f_i,f_j\}df_1\wedge\ldots d\hat{f_i}\ldots d\hat{f_j}\ldots\wedge df_k\,.\end{aligned}$$

Proof. Equations (21) and (22) are proven in [Bry88, §2], and equation (20) in [Bry90, §5]. ☐

Remark 2.2. The special cases $k = 1$ and $k = 2$ of (22) will be of special use later on:

$$\delta(f_0df_1) = \{f_0, f_1\}\,, \tag{23}$$

$$\delta(f_0df_1 \wedge df_2) = \{f_0, f_1\}df_2 - \{f_0, f_2\}df_1 - f_0d\{f_1, f_2\}\,. \tag{24}$$

Lemma 2.3. *$d\Omega^0(X) \subseteq \delta\Omega^2(X)$.*

Proof. For $f \in \Omega^0(X)$, we have $*df = df \wedge \omega^{n-1}/(n-1)!$ by eqn. (20), so with $*^2 = \mathrm{Id}$, we find $df = *d(f\omega^{n-1}/(n-1)!)$. Using $\delta = (-1)^{k+1} * d\, *$, we then find that $df = -\delta(*f\omega^{n-1}/(n-1)!)$ is in $\delta\Omega^2(X)$ as desired. $\square$

We define the compactly supported *canonical homology* $H^{\mathrm{can}}_{c,\bullet}(X)$ to be the homology of the complex $(\Omega^\bullet_c(X), \delta)$. By definition then, the star operator $* : \Omega^k_c(X) \to \Omega^{2n-k}_c(X)$, adorned with the sign $(-1)^{k(k-1)/2}$, defines an isomorphism of chain complexes

$$\begin{tikzcd}
\Omega^0_c(X) \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \Omega^1_c(X) \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \dots \arrow[r, "d"] & \Omega^{2n-1}_c(X) \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \Omega^{2n}_c(X) \arrow[d, leftrightarrow, "*"] \\
\Omega^{2n}_c(X) \arrow[r, "\delta"] & \Omega^{2n-1}_c(X) \arrow[r, "\delta"] & \dots \arrow[r, "\delta"] & \Omega^1_c(X) \arrow[r, "\delta"] & \Omega^0_c(X),
\end{tikzcd}$$

hence an isomorphism $H^k_{\mathrm{dR},c}(X) \simeq H^{\mathrm{can}}_{c,2n-k}(X)$. Mutatis mutandis, the same argument in the noncompactly supported case yields an isomorphism $H^k_{\mathrm{dR}}(X) \simeq H^{\mathrm{can}}_{2n-k}(X)$.

Proposition 2.4. *The symplectic Hodge star operator yields isomorphisms $H^k_{\mathrm{dR}}(X) \simeq H^{\mathrm{can}}_{2n-k}(X)$ and $H^k_{\mathrm{dR},c}(X) \simeq H^{\mathrm{can}}_{c,2n-k}(X)$.*

Remark 2.5. For an arbitrary Poisson manifold, the canonical homology $H^{\mathrm{can}}_\bullet(X)$ is defined as the homology of the complex $\Omega^\bullet(X)$ with respect to the differential $\delta = i_\pi \circ d - d \circ i_\pi$. Brylinski's theorem 2.1 then shows that in case X is symplectic, $H^{\mathrm{can}}_k(X) \simeq H^{2n-k}_{\mathrm{dR}}(X)$.

2.3 Lie algebra cohomology

The objective of this paper is to classify central extensions of certain locally convex Lie algebras.

Locally convex Lie algebras. A *locally convex Lie algebra* is a locally convex topological vector space $\mathfrak{g}$, together with a continuous Lie bracket $[\,\cdot\,,\,\cdot\,] \colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$.

Let (X, ω) be a symplectic manifold, which we will always assume to be second countable. If $K \subseteq X$ is a compact subset with open interior, then

$$C^\infty_K(X) := \{f \in C^\infty(X)\,;\, \mathrm{supp}(f) \subseteq K\}$$

and

$$\mathrm{vec}_K(X) := \{v \in \mathrm{vec}(X)\,;\, \mathrm{supp}(v) \subseteq K\}$$

are locally convex Lie algebras for the Fréchet topology of uniform convergence in all derivatives. We equip $C^\infty(X)$ and $\mathrm{vec}(X)$ with the Fréchet topology that comes from the inverse limit over $K \subseteq X$, and $C^\infty_c(X)$ with the LF-topology that comes from the (strict) direct limit. The former makes $\mathrm{sp}(X)$, $\mathrm{ham}(X)$ and $C^\infty(X)$ into locally convex (Fréchet) Lie algebras, and the latter makes $C^\infty_c(X)$ and $C^\infty_{c,0}(X)$ into locally convex (LF) Lie algebras [Trè67, §I.13].

Definition 2.6. A *continuous central extension* $(\widehat{\mathfrak{g}}, \iota, \pi)$ of a locally convex Lie algebra $\mathfrak{g}$ by a locally convex vector space $\mathfrak{a}$ is a continuous exact sequence

$$0 \to \mathfrak{a} \xrightarrow{\iota} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0$$

of locally convex Lie algebras such that $\iota(\mathfrak{a}) \subseteq \widehat{\mathfrak{g}}$ is central. It is called *linearly split* if there exists a continuous linear map $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ such that $\pi \circ \sigma = \mathrm{Id}_{\mathfrak{g}}$, and *split* if σ is a Lie algebra homomorphism. An *isomorphism* between $(\widehat{\mathfrak{g}}, \iota, \pi)$ and $(\widehat{\mathfrak{g}}', \iota', \pi')$ is a continuous Lie algebra isomorphism $\phi\colon \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}'$ such that $\phi \circ \pi' = \pi$ and $\phi \circ \iota = \iota'$.

Remark 2.7. Continuous central extensions of $\mathfrak{g}$ by $\mathbb{R}$ are automatically linearly split. Indeed, the map $\iota^{-1}\colon \iota(\mathbb{R}) \to \mathbb{R}$ extends to a continuous linear functional $\gamma\colon \widehat{\mathfrak{g}} \to \mathbb{R}$ by the Hahn-Banach theorem for locally convex vector spaces [Rud91, Thm. 3.6], and σ is the inverse of $\pi\colon \mathrm{Ker}(\gamma) \to \mathfrak{g}$.

Linearly split continuous central extensions $\mathfrak{a} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ are the infinitesimal counterpart of central extensions $A \to \widehat{G} \to G$ of infinite dimensional Lie groups modelled on locally convex spaces [Mil84, Nee06]. Continuity of the Lie algebra homomorphisms comes from smoothness of the group homomorphisms, and a splitting of $\widehat{\mathfrak{g}} \to \mathfrak{g}$ exists because, by assumption, $\widehat{G} \to G$ is a locally trivial principal bundle [Nee02b].

Central extensions are classified by second Lie algebra cohomology. In order to classify *continuous* central extensions, we will need *continuous* Lie algebra cohomology.

Definition 2.8. The *continuous Lie algebra cohomology* $H^n(\mathfrak{g}, \mathbb{R})$ of a locally convex Lie algebra $\mathfrak{g}$ is the cohomology of the complex $C^\bullet(\mathfrak{g}, \mathbb{R})$, where $C^n(\mathfrak{g}, \mathbb{R})$ is the space of continuous alternating n-linear maps $\psi\colon \mathfrak{g}^n \to \mathbb{R}$ with differential $\mathtt{d}\colon C^n(\mathfrak{g}, \mathbb{R}) \to C^{n+1}(\mathfrak{g}, \mathbb{R})$ defined by zero on $C^0(\mathfrak{g}, \mathbb{R})$, and

$$\mathtt{d}\psi(x_0, \ldots, x_n) := \sum_{0 \leq i < j \leq n} (-1)^{i+j} \omega([x_i, x_j], x_0, \ldots, \widehat{x}_i, \ldots, \widehat{x}_j, \ldots, x_n)$$

for $n \geq 1$.

The cohomology in degree 1 classifies the continuous characters of $\mathfrak{g}$, that is, the continuous Lie algebra homomorphisms $\mathfrak{g} \to \mathbb{R}$. Indeed, $H^1(\mathfrak{g}, \mathbb{R})$ is the topological dual of the abelian Lie algebra $(\mathfrak{g}/\overline{[\mathfrak{g}, \mathfrak{g}]})$, where $\overline{[\mathfrak{g}, \mathfrak{g}]}$ is the closure of the commutator ideal. In particular, a locally convex Lie algebra is topologically perfect ($\mathfrak{g} = \overline{[\mathfrak{g}, \mathfrak{g}]}$) if and only if $H^1(\mathfrak{g}, \mathbb{R})$ vanishes.

The following standard result (cf. [JN15]) interprets $H^2(\mathfrak{g}, \mathbb{R})$ in terms of central extensions.

Proposition 2.9. *Up to isomorphism, the continuous central extensions of the locally convex Lie algebra $\mathfrak{g}$ are classified by $H^2(\mathfrak{g}, \mathbb{R})$.*

Proof. Given a 2-cocycle $\psi\colon \mathfrak{g}^2 \to \mathbb{R}$, we define the Lie algebra $\widehat{\mathfrak{g}}_\psi$ by

$$\widehat{\mathfrak{g}}_\psi := \mathbb{R} \oplus_\psi \mathfrak{g}$$

with the Lie bracket $[(z, x), (z', x')] := \big(\psi(x, x'), [x, x']\big)$. Equipped with the obvious maps $\mathbb{R} \to \widehat{\mathfrak{g}}_\psi \to \mathfrak{g}$, this is a continuous central extension of $\mathfrak{g}$ by $\mathbb{R}$, and a cohomologous cocycle $\psi' = \psi + \mathtt{d}\lambda$ gives rise to an isomorphic Lie algebra $\widehat{\mathfrak{g}}_{\psi'}$, where the isomorphism $\widehat{\mathfrak{g}}_\psi \to \widehat{\mathfrak{g}}_{\psi'}$ is given by $(z, x) \mapsto (z - \lambda(x), x)$. Conversely, every continuous central extension $(\widehat{\mathfrak{g}}, \iota, \pi)$ admits a continuous linear splitting $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$ with $\pi \circ \sigma = \mathrm{Id}_{\mathfrak{g}}$ by the Hahn-Banach Theorem, cf. Remark 2.7. The

continuous central extension $\mathbb{R} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$ is thus isomorphic to $\mathbb{R} \to \widehat{\mathfrak{g}}_\psi \to \mathfrak{g}$ with

$$\psi(x, x') := \gamma\Big([\sigma(x), \sigma(x')] - \sigma([x, x'])\Big) . \tag{25}$$

The isomorphism $\widehat{\mathfrak{g}}_\psi \to \widehat{\mathfrak{g}}$ is given by $(z, x) \mapsto \sigma(x) + \iota(z)$. □

If $(\widehat{\mathfrak{g}}, \iota, \pi)$ is split, there exists a continuous character $\lambda\colon \widehat{\mathfrak{g}} \to \mathbb{R}$ (a *splitting*) such that $\lambda \circ \iota = \mathrm{Id}_{\mathbb{R}}$. The inverse of $\pi\colon \mathrm{Ker}(\lambda) \to \mathfrak{g}$ then yields a continuous Lie algebra homomorphism $\sigma\colon \mathfrak{g} \to \widehat{\mathfrak{g}}$, so that $(\widehat{\mathfrak{g}}, \iota, \pi)$ is isomorphic to the trivial extension, and its class in $H^2(\mathfrak{g}, \mathbb{R})$ is zero. The space

$$\{\lambda \in \widehat{\mathfrak{g}}' \,;\, \mathtt{d}\lambda = 0, \lambda \circ \iota = \mathrm{Id}_{\mathbb{R}}\}$$

of continuous splittings is then an affine subspace of $H^1(\widehat{\mathfrak{g}}, \mathbb{R})$.

Definition 2.10. If $\mathfrak{g}$ is a locally convex Lie algebra, and $\mathfrak{a}$ a locally convex space, then a continuous central extension

$$\mathfrak{z} \to \widehat{\mathfrak{g}} \to \mathfrak{g}$$

of $\mathfrak{g}$ is called $\mathfrak{a}$-*universal* if for every linearly split central extension $\mathfrak{a} \to \mathfrak{g}^\sharp \to \mathfrak{g}$ of $\mathfrak{g}$ by $\mathfrak{a}$, there exists a unique continuous Lie algebra homomorphism $\phi\colon \widehat{\mathfrak{g}} \to \mathfrak{g}^\sharp$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathfrak{z} \arrow[r] \arrow[d, "\phi|_{\mathfrak{z}}"] & \widehat{\mathfrak{g}} \arrow[r] \arrow[d, "\phi"] & \mathfrak{g} \arrow[d, "\mathrm{Id}"] \\
\mathfrak{a} \arrow[r] & \mathfrak{g}^\sharp \arrow[r] & \mathfrak{g}.
\end{tikzcd}$$

A central extension is called *universal* if it is linearly split and $\mathfrak{a}$-universal for every locally convex space $\mathfrak{a}$.

If a universal central extension exists, then it is unique up to continuous isomorphisms. A sufficient condition for existence is that $\mathfrak{g}$ be a perfect Fréchet Lie algebra with finitely generated $H^2(\mathfrak{g}, \mathbb{R})$ (cf. [Nee02b, Cor. 2.13]).

2.4 Some useful lemmas

We close this introductory section with two small lemmas that will be useful throughout the text.

Lemma 2.11. *If X is noncompact, then $\delta : \Omega^1(X) \to C^\infty(X)$ is surjective. If X is compact, then* $\mathrm{Im}(\delta) = C^\infty_{c,0}(X)$.

Proof. The cokernel of $\delta\colon \Omega^1(X) \to C^\infty(X)$ is by definition $H_0^{\mathrm{can}}(X)$, hence isomorphic to $H^{2n}_{\mathrm{dR}}(X)$ (cf. Prop. 2.4). If X is noncompact, then $H^{2n}_{\mathrm{dR}}(X)$ vanishes, so δ is surjective. If X is compact, the inclusion $\mathrm{Im}(\delta) \subseteq C^\infty_{c,0}(X)$ holds because for all $\alpha \in \Omega^1(X)$, we have $\int_X (\delta\alpha)\omega^n/n! = \int_X *\delta\alpha = \int_X d * \alpha = 0$. Equality follows because the cokernel $H_0^{\mathrm{can}}(X) \simeq H^{2n}_{\mathrm{dR}}(X)$ of δ is 1-dimensional. □

Lemma 2.12. *For $\alpha \in \Omega^1(X)$ and v_1, v_2, v_3 and v in* $\mathrm{vec}(X)$*, we have*

$$\alpha(v)\omega^n/n! \;=\; \alpha \wedge i_v\omega \wedge \omega^{n-1}/(n-1)! \,, \tag{26}$$

$$\sum_{\mathrm{cycl}} \alpha(v_1)\omega(v_2, v_3)\omega^n/n! \;=\; \alpha \wedge i_{v_1}\omega \wedge i_{v_2}\omega \wedge i_{v_3}\omega \wedge \omega^{n-2}/(n-2)! \,. \tag{27}$$

Proof. The proof is a simple computation. Equation (26) is obtained from expanding $i_v(\alpha \wedge \omega^n)/n! = 0$. Applying this to $\alpha = -i_w\omega$, we find

$$\omega(v,w)\omega^n/n! = i_v\omega \wedge i_w\omega \wedge \omega^{n-1}/n!\,. \tag{28}$$

Equation (27) is then derived by expanding

$$0 = i_{v_3}(\alpha \wedge i_{v_1}\omega \wedge i_{v_2}\omega \wedge \omega^{n-1}/(n-1)!)$$

into an alternating sum of four terms, and applying (26) and (28) repeatedly. □

3 Characters

In order to classify the continuous central extensions of the Poisson Lie algebra and the Lie algebra of Hamiltonian vector fields (which, by Prop. 2.9, is equivalent to determining the second Lie algebra cohomology), we will need the first continuous Lie algebra cohomology. This is equivalent to finding the closure of the commutator ideal, or, equivalently, the set of continuous characters $\mathfrak{g} \to \mathbb{R}$.

3.1 Characters of $C^\infty_{c,0}(X)$ and $C^\infty_c(X)$

For the following proposition and its proof, we follow closely the paper [ALDM74] of Avez, Lichnerowicz and Diaz-Miranda.

Define the linear functional

$$\lambda\colon C^\infty_c(X) \to \mathbb{R}, \quad \lambda(f) := \int_X f\omega^n/n!, \tag{29}$$

and note that the ideal defined in (17) is $C^\infty_{c,0}(X) = \mathrm{Ker}(\lambda)$.

Proposition 3.1. ([ALDM74, §12]) *The commutator ideal of $C^\infty_c(X)$ is the perfect Lie algebra $C^\infty_{c,0}(X)$,*

$$[C^\infty_c(X), C^\infty_c(X)] = [C^\infty_{c,0}(X), C^\infty_{c,0}(X)] = C^\infty_{c,0}(X)\,.$$

Proof. Suppose $f \in C^\infty_c(X)$ is a commutator, $f = \{g,h\}$ for $g,h \in C^\infty_c(X)$. We show that there exist $g_0, h_0 \in C^\infty_{c,0}(X)$ with $f = \{g_0, h_0\}$. Choose $\chi \in C^\infty_c(X)$ which is constant on $\mathrm{supp}(g) \cup \mathrm{supp}(h)$ and such that $\lambda(\chi) = 1$, with λ defined in (29). The functions $g_0 := g - \lambda(g)\chi$ and $h_0 := h - \lambda(h)\chi$ are then the elements of $C^\infty_{c,0}(X)$ with the desired relation $\{g_0, h_0\} = \{g, h\}$, because $\{g,\chi\} = \{h,\chi\} = 0$. By applying this to (finite) sums of commutators, we see that $[C^\infty_c(X), C^\infty_c(X)]$ equals $[C^\infty_{c,0}(X), C^\infty_{c,0}(X)]$.

Because $L_{X_g}\omega = 0$ and $\{g,h\} = L_{X_g}h$, we have $f\omega^n/n! = L_{X_g}(h\omega^n/n!)$, and hence $f\omega^n/n! = d(hi_{X_g}\omega^n/n!)$. This shows that $f = \{g,h\}$ is an element of $C^\infty_{c,0}(X)$. Since this argument extends to finite sums of commutators, we find

$$[C^\infty_c(X), C^\infty_c(X)] = [C^\infty_{c,0}(X), C^\infty_{c,0}(X)] \subseteq C^\infty_{c,0}(X)\,,$$

and we record the useful equation

$$\begin{aligned} \{g,h\}\omega^n/n! &= d(hi_{X_g}\omega^n/n!) \\ &= -d(hdg \wedge \omega^{n-1}/(n-1)!) \\ &= dg \wedge dh \wedge \omega^{n-1}/(n-1)! \end{aligned} \tag{30}$$

for later use.

For the converse inclusion, suppose that $f\omega^n/n! = d\psi$ with ψ compactly supported. We show that X_f is in the commutator ideal. Write $\psi = \sum_{k=1}^m \psi_k$, where ψ_k has compact support in an area with Darboux coordinates x^i, p^i. Note that $dx^i \wedge \omega^{n-1}/(n-1)!$ and $dp^i \wedge \omega^{n-1}/(n-1)!$ constitute a basis for $\wedge^{2n-1}TX_x$ at each point $x \in X$, so that we can write

$$\psi_k = \sum_{i=1}^n \phi_k^i dx^i \wedge \omega^{n-1}/(n-1)! + \chi_k^i dp^i \wedge \omega^{n-1}/(n-1)!\,,$$

with ϕ_k^i and χ_k^i compactly supported. Then choose compactly supported ξ_k^i and η_k^i that equal x^i and p^i on the support of ϕ_k^i and χ_k^i respectively to obtain

$$\psi_k = \sum_{i=1}^n \phi_k^i d\xi_k^i \wedge \omega^{n-1}/(n-1)! + \chi_k^i d\eta_k^i \wedge \omega^{n-1}/(n-1)!\,,$$

and thus

$$d\psi = \sum_{k=1}^m \sum_{i=1}^n d\phi_k^i \wedge d\xi_k^i \wedge \omega^{n-1}/(n-1)! + d\chi_k^i \wedge d\eta_k^i \wedge \omega^{n-1}/(n-1)!\,.$$

By eqn. (30), $d\psi = f\omega^n/n!$ then implies

$$f = \sum_{i=1}^n \sum_{k=1}^m \{\phi_k^i, \xi_k^i\} + \{\chi_k^i, \eta_k^i\}\,.$$

This shows the inverse inclusion $[C_c^\infty(X), C_c^\infty(X)] \supseteq C_{c,0}^\infty(X)$. ☐

Since $C_{c,0}^\infty(X)$ is perfect, it is in particular topologically perfect, so its first continuous Lie algebra cohomology vanishes. Since $C_{c,0}^\infty(X)$ is closed in $C_c^\infty(X)$, the first cohomology $H^1(C_c^\infty(X), \mathbb{R})$ is equal to $(C_c^\infty(X)/C_{c,0}^\infty(X))'$. The volume form $\omega^n/n!$ then yields an isomorphism with the topological dual $H_{\mathrm{dR},c}^{2n}(X)' = (\Omega_c^{2n}(X,\mathbb{R})/d\Omega_c^{2n-1}(X,\mathbb{R}))'$ of the compactly supported de Rham cohomology, taking $\gamma \in H_{\mathrm{dR},c}^{2n}(X)'$ to the class of the cocycle $\lambda_\gamma(f) = \gamma([f\omega^n/n!])$. If X has finitely many connected components, then the map $H_{2n}(X,\mathbb{R}) \to H_{\mathrm{dR},c}^{2n}(X)'$ defined by $[X_0] \mapsto \gamma_{X_0}$ with $\gamma_{X_0}([\beta]) := \int_{[X_0]} \beta$ is an isomorphism. The image of the integral singular cohomology $H_{2n}(X,\mathbb{Z})$ then yields a lattice in $H^1(C_c^\infty(X),\mathbb{R})$ with generators $[\lambda_{X_0}]$, where

$$\lambda_{X_0}(f) := \int_{X_0} f\omega^n/n! \tag{31}$$

and $[X_0]$ runs over the connected components of X.

Corollary 3.2. *The continuous first Lie algebra cohomology $H^1(C_{c,0}^\infty(X),\mathbb{R})$ vanishes. Furthermore, $H^1(C_c^\infty(X),\mathbb{R})$ is isomorphic to $H_{\mathrm{dR},c}^{2n}(X)'$. If X has finitely many connected components, then this is isomorphic to $H_{2n}(X,\mathbb{R})$.*

3.2 Characters of $C^\infty(X)$

We turn to the Poisson Lie algebra $C^\infty(X)$ without compact support condition. It is the direct product

$$C^\infty(X) = \prod C^\infty(X_x),$$

where the product runs over the connected components X_x of X. If X_x is compact, then by Corr. 3.2, the pullback of the cocycle λ_{X_x} by the canonical projection $C^\infty(X) \to C^\infty(X_x)$ contributes to the first cohomology. The following proposition says that there is no such contribution if X_x is noncompact.

Proposition 3.3. *The first Lie algebra cohomology $H^1(C^\infty(X), \mathbb{R})$ of the Poisson Lie algebra $C^\infty(X)$ is isomorphic to $H_{2n}(X, \mathbb{R})$. Moreover, $C^\infty(X)$ is perfect, $C^\infty(X) = [C^\infty(X), C^\infty(X)]$, if and only if X has no compact connected components.*

Proof. Let $X = X_{\mathrm{cpt}} \sqcup X_{\mathrm{ncpt}}$, with X_{cpt} the union of the compact connected components X_x, and $X_{\mathrm{ncpt}} := X - X_{\mathrm{cpt}}$. By Corrolary 3.2, any continuous character $\lambda \colon C^\infty(X) \to \mathbb{R}$ restricts to a multiple of λ_{X_x} (cf. eqn. (31)) on X_x. By continuity, λ must then be a *finite* linear combination of λ_{X_x} on $C^\infty(X_{\mathrm{cpt}}, \mathbb{R})$. Since $C^\infty_{c,0}(X_{\mathrm{ncpt}})$ is dense in $C^\infty(X_{\mathrm{ncpt}})$ for the topology of uniform convergence of all derivatives on compact subsets, it follows from Prop. 3.1 that $C^\infty(X_{\mathrm{ncpt}})$ is topologically perfect, so that the linear map

$$H_{2n}(X, \mathbb{R}) \to H^1(C^\infty(X), \mathbb{R})$$

defined by $[X_x] \mapsto \lambda_{X_x}$ is an isomorphism. All that is left to show is that $C^\infty(X)$ is perfect (rather than merely *topologically* perfect) if X has no compact connected components. For this, we use the Brouwer-Lebesgue 'Paving Principle'. Since $H^{2n}_{\mathrm{dR}}(X) = 0$, there exists a $\psi \in \Omega^{2n-1}(X)$ such that $f\omega^n = d\psi$. Find a cover $U_{k,r}$ of X where $k \in \mathbb{N}$ is a countable index, $r \in \{1, \ldots, 2n+1\}$ is a finite index, and $U_{k,r} \cap U_{k',r} = \emptyset$ for all $k \neq k'$. (Such a cover exists [HW41, Thm. V1].) As in the proof of Prop. 3.1, we can write $\psi = \sum_{k=1}^{\infty} \sum_{r=1}^{2n+1} \psi_{k,r}$ with $\psi_{k,r}$ supported in $U_{k,r}$. (Evaluated in a single point, the sum has at most $2n+1$ nonzero terms.) We define $f_{k,r}$ by $d\psi_{k,r} = f_{k,r}\,\omega^n/n!$, and follow the proof of Prop. 3.1 to find $2n$ functions $g^i_{k,r}$ and $h^i_{k,r}$, supported in $U_{k,r}$, that satisfy $f_{k,r} = \sum_{i=1}^{2n} \{g^i_{k,r}, h^i_{k,r}\}$. We set $G^i_r := \sum_{k=1}^{\infty} g^i_{k,r}$ and $H^i_r := \sum_{k=1}^{\infty} h^i_{k,r}$, and use the fact that $\{g^i_{k,r}, h^i_{k',r}\} = 0$ for $k \neq k'$ to find

$$\sum_{r=1}^{2n+1} \sum_{i=1}^{2n} \{G^i_r, H^i_r\} = \sum_{r=1}^{2n+1} \sum_{i=1}^{2n} \sum_{k=1}^{\infty} \{g^i_{k,r}, h^i_{k,r}\} = \sum_{r=1}^{2n+1} \sum_{k=1}^{\infty} f_{k,r} = f\,.$$

□

Remark 3.4. Note that if X has no compact connected components, then $f \in C^\infty(X)$ can be written as a sum of at most $2n(2n+1)$ commutators. By the above reasoning, the same holds for $f \in C^\infty_{c,0}(X)$ if X is connected and compact.

We summarise the classification of the characters of $C^\infty_{c,0}(X)$, $C^\infty_c(X)$ and $C^\infty(X)$. Since $C^\infty_{c,0}(X)$ is perfect by Prop. 3.1, it has no nontrivial characters, continuous or not. If X_x is a connected component of X, then $\lambda_{X_x}(f) =$

$\int_{X_x} f\omega^n/n!$ defines a character on $C^\infty_c(X)$. If we specify a real number $a_x \in \mathbb{R}$ for every connected component $X_x \subseteq X$, then

$$\lambda = \sum a_x \lambda_{X_x} \tag{32}$$

is a character of $C^\infty_c(X)$. If X_x is compact, then λ_{X_x} extends to a character of $C^\infty(X)$, and the characters of $C^\infty(X)$ are *finite* linear combinations of the λ_{X_x}. The characters of $C^\infty(X)$ and $C^\infty_c(X)$ are automatically continuous.

3.3 The Kostant-Souriau extension

If X is a connected symplectic manifold, then the Lie algebra homomorphism $\pi\colon C^\infty(X) \to \mathrm{ham}(X)$ defined by $f \mapsto X_f$ has kernel $\mathbb{R}\mathbf{1}$. The continuous central extension

$$\mathbb{R}\mathbf{1} \xrightarrow{\iota} C^\infty(X) \xrightarrow{\pi} \mathrm{ham}(X), \tag{33}$$

called the *Kostant-Souriau extension*, plays a central role in geometric quantization [Kos70, Sou70]. Evaluation in $x \in X$ is a continuous linear map $\mathrm{ev}_x\colon C^\infty(X) \to \mathbb{R}$ with $\mathrm{ev}_x \circ \iota = \mathrm{Id}_{\mathbb{R}}$, and the corresponding linear map $\mathrm{ham}(X) \to C^\infty(X)$ maps X_f to $f - f(x)$. This induces via (25) the *Kostant-Souriau cocycle*

$$\psi_{\mathrm{KS}}(X_f, X_g) := \{f, g\}_x. \tag{34}$$

The class $[\psi_{\mathrm{KS}}] \in H^2(\mathrm{ham}(X), \mathbb{R})$ is independent of $x \in X$ by Prop. 2.9.

Remark 3.5 (Hamiltonian actions)**.** Let G be a finite dimensional Lie group with Lie algebra $\mathfrak{g}$ and X a connected symplectic manifold. An action $A\colon G \to \mathrm{Diff}(X)$ of G on X is called *symplectic* if it preserves ω, and *weakly hamiltonian* (cf. [MS98, §5.2]) if the image of the infinitesimal action $a\colon \mathfrak{g} \to \mathrm{sp}(X)$ lies in $\mathrm{ham}(X) \subseteq \mathrm{sp}(X)$. A *comomentum map* is a linear splitting of (33) along $a\colon \mathfrak{g} \to \mathrm{ham}(X)$, that is, a linear map $J\colon \mathfrak{g} \to C^\infty(X)$ such that $a(\xi) = X_{J(\xi)}$ for all $\xi \in \mathfrak{g}$. It determines a *momentum map* $\mu\colon X \to \mathfrak{g}^*$ by $\mu(x)(\xi) = J(\xi)(x)$. We then have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, hook] & C^\infty(X) \arrow[r, two heads] & \mathrm{ham}(X) \arrow[r, hook] & \mathrm{sp}(X) \arrow[r, two heads] & H^1_{\mathrm{dR}}(X) \\
 & & & \mathfrak{g} \arrow[ull, "J"] \arrow[ul, "a"] \arrow[u, "a"'] \arrow[ur, "0"'] &
\end{tikzcd} \tag{35}$$

The action is called *hamiltonian* if there exists a comomentum map $J\colon \mathfrak{g} \to C^\infty(X)$ which is a Lie algebra homomorphism. The failure of J to be a homomorphism is measured by the 2-cocycle $\psi_J\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$, defined (cf. [Sou70, p. 109]) by

$$\psi_J(\xi, \eta) = \{J(\xi), J(\eta)\} - J([\xi, \eta])\,. \tag{36}$$

Since X is connected, the difference $c = J' - J$ between two comomentum maps is a 1-cochain $c\colon \mathfrak{g} \to \mathbb{R}$, and $\psi_{J'} - \psi_J = \delta c$. It follows that the class $[\psi_J] \in H^2(\mathfrak{g}, \mathbb{R})$ does not depend on the choice of J, and is zero if and only if the action is hamiltonian. The Kostant-Souriau class $[\psi_{KS}]$ is *universal* for these non-equivariance classes, in the sense that $[\psi_J] \in H^2(\mathfrak{g}, \mathbb{R})$ is the pullback of $[\psi_{KS}] \in H^2(\mathrm{ham}(X), \mathbb{R})$ along the infinitesimal action $a\colon \mathfrak{g} \to \mathrm{ham}(X)$.

The following corollary shows that $[\psi_{\mathrm{KS}}] = 0$ if and only if X is compact. In particular, every weakly hamiltonian action on a compact, connected, symplectic manifold is hamiltonian.

Corollary 3.6. *The Kostant-Souriau extension is split if and only if X is compact. The splitting $\langle\lambda_X\rangle\colon C^\infty(X) \to \mathbb{R}$ is then given by*

$$\langle\lambda_X\rangle(f) = \frac{1}{\mathrm{vol}(X)} \int_X f\omega^n/n!, \tag{37}$$

with $\mathrm{vol}(X) := \int_X \omega^n/n!$ *the symplectic volume of X.*

Proof. If X is compact, then $\langle\lambda_X\rangle$ is a continuous splitting; it is a multiple of the continuous character λ_X of eqn. (31), and it manifestly satisfies $\langle\lambda_X\rangle \circ \iota = \mathrm{Id}_{\mathbb{R}}$. If X is noncompact, there are no nontrivial characters, continuous or not, since $C^\infty(X)$ is perfect by Prop. 3.3, so the extension (33) is not split. $\square$

If X is compact connected, we thus have an isomorphism of $\mathrm{ham}(X)$ with $\mathrm{Ker}(\langle\lambda_X\rangle) = C^\infty_{c,0}(X)$, and $C^\infty(X) \simeq C^\infty_{c,0}(X) \oplus \mathbb{R}$.

Corollary 3.7. ([ALDM74, §12]) *Let X be a symplectic manifold. Then the Lie algebra* $\mathrm{ham}(X)$ *of hamiltonian vector fields is perfect. In particular, the first Lie algebra cohomology* $H^1(\mathrm{ham}(X), \mathbb{R})$ *is trivial.*

Proof. If a connected component X_x of X is compact, then $\mathrm{ham}(X_x) \simeq C^\infty_{c,0}(X)$ is perfect by Prop. 3.1. If X_x is noncompact, then $\mathrm{ham}(X_x)$ is perfect as the homomorphic image of the perfect (Prop. 3.3) Lie algebra $C^\infty(X_x)$. When restricted to X_x, the hamiltonian vector field X_f can be written as a sum of at most $2n(2n+1)$ commutators $[X^x_{G^i_r}, X^x_{H^i_r}]$, cf. Rk. 3.4. Since $X^x_{G^i_r}$ and $X^{x'}_{H^i_r}$ commute if they live on different connected components, we can write X_f as the sum of at most $2n(2n+1)$ commutators $\sum_{X_x} X^x_{G^i_r}$ and $\sum_{X_x} X^x_{H^i_r}$. $\square$

4 Continuous central extensions

Having determined the characters of the Lie algebras at hand, we now turn to the heart of the paper: classifying the continuous central extensions of the compactly supported Poisson algebra $C^\infty_c(X)$. From this, we derive a similar classification for $C^\infty(X)$ and $\mathrm{ham}(X)$.

4.1 Diagonal cocycles

The first step is to show that the Lie algebras of interest to us have *diagonal* cocycles. Lie algebra cohomology with diagonal cocycles is extensively developed in the monograph [Fuk86].

Definition 4.1. A 2-cocycle ψ on a Lie subalgebra $\mathfrak{g}$ of $C^\infty(X)$ or $\mathrm{vec}(X)$ is called *diagonal* if $\psi(f,g) = 0$ whenever $\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset$.

The easiest case is the perfect Lie algebra $\mathfrak{g} = C^\infty_{c,0}(X)$.

Proposition 4.2. *Let X be a symplectic manifold. Then every 2-cocycle on $C^\infty_{c,0}(X)$, continuous or not, is diagonal.*

Proof. Let $f, g \in C^\infty_{c,0}(X)$ be such that f and g have disjoint support. Because X is a normal space, one can choose disjoint open sets $U, V \subset X$ with $\mathrm{supp}(f) \subset U$ and $\mathrm{supp}(g) \subset V$. According to Lemma 3.1 applied to $C^\infty_{c,0}(V)$, one can write g as a sum of commutators $g = \sum_{i=1}^n \{\phi_i, \xi_i\}$ where ϕ_i and ξ_i are contained in $C^\infty_{c,0}(V)$ and whose support is therfore disjoint with that of df. Using the cocycle identity for ψ and the fact that f commutes with ϕ_i and ξ_i, we find

$$\psi(f,g) = \sum_{i=1}^n \psi(f, \{\phi_i, \xi_i\}) = -\sum_{i=1}^n \psi(\phi_i, \{\xi_i, f\}) + \psi(\xi_i, \{f, \phi_i\}) = 0\,. \quad (38)$$

This proves that every cocycle on $C^\infty_{c,0}(X)$ is diagonal. □

Building on the above trick, we derive the corresponding result for the Poisson Lie algebra.

Proposition 4.3. *For a connected symplectic manifold X, every 2-cocycle on $C^\infty_c(X)$ and $C^\infty(X)$, continuous or not, is diagonal.*

Proof. We start with $C^\infty(X)$ for X noncompact. For $f, g \in C^\infty(X)$ with disjoint support, we find disjoint open sets $U, V \subseteq X$ with $\mathrm{supp}(f) \subset U$ and $\mathrm{supp}(g) \subset V$. Since V has no compact connected components, we can apply Lemma 3.3 to $C^\infty(V)$ to write g as a finite sum of commutators with support in V, and follow the proof of Prop. 4.2 to conclude that every 2-cocycle on $C^\infty(X)$ is diagonal.

Next we pass to $C^\infty_c(X)$ for X noncompact. Let ψ be a 2-cocycle on $C^\infty_c(X)$. For $f \in C^\infty_c(X)$ and $g \in C^\infty_{c,0}(X)$ with disjoint support, the proof of equation (38) can be followed word by word to show that $\psi(f,g) = 0$. For $f, g \in C^\infty_c(X)$, we now define

$$\widetilde{\psi}(f,g) := \psi(f, g - \Delta_g)\,,$$

where $\Delta_g \in C^\infty_c(X)$ satisfies $\int_X \Delta_g \omega^n = \int_X g\omega^n$ (so that $g - \Delta_g \in C^\infty_{c,0}(X)$) and $\mathrm{supp}(f) \cap \mathrm{supp}(\Delta_g) = \emptyset$. This does not depend on the choice of Δ_g; another choice Δ'_g yields $\Delta'_g - \Delta_g \in C^\infty_{c,0}(X)$, and since $\mathrm{supp}(f)$ is disjoint from $\mathrm{supp}(\Delta'_g - \Delta_g)$, we have $\psi(f, \Delta'_g - \Delta_g) = 0$. Note that if f and g have disjoint support, then we may choose $g = \Delta_g$, so that $\widetilde{\psi}(f,g) = 0$, i.e. $\widetilde{\psi}$ is diagonal.

We show that $\widetilde{\psi}$ is equal to ψ. The bilinear map

$$\psi - \widetilde{\psi} \colon C^\infty_c(X) \times C^\infty_c(X) \to \mathbb{R}$$

vanishes on $C^\infty_c(X) \times C^\infty_{c,0}(X)$ by definition, and it vanishes on $C^\infty_{c,0}(X) \times C^\infty_c(X)$ because for $f_0 \in C^\infty_{c,0}(X)$ and $g \in C^\infty_c(X)$, disjointness of $\mathrm{supp}(f_0)$ and $\mathrm{supp}(\Delta_g)$ implies $\psi(f_0, \Delta_g) = 0$, hence $\widetilde{\psi}(f_0, g) = \psi(f_0, g)$. It follows that $\psi - \widetilde{\psi}$ factors through a bilinear map on $C^\infty_c(X)/C^\infty_{c,0}(X)$. This map is antisymmetric because any two classes $[f], [g] \in C^\infty_c(X)/C^\infty_{c,0}(X)$ have representatives with disjoint support, so that $\widetilde{\psi}(f,g) = 0$ and $(\psi - \widetilde{\psi})([f],[g]) = \psi(f,g) = -\psi(g,f) = -(\psi - \widetilde{\psi})([g],[f])$. Since X is connected, $C^\infty_c(X)/C^\infty_{c,0}(X)$ is 1-dimensional, so $\psi - \widetilde{\psi}$ is zero. It follows that $\psi = \widetilde{\psi}$ is diagonal.

It remains to show that for X compact connected, every 2-cocycle ψ on $C^\infty(X)$ is diagonal. Let f and g have disjoint nonempty support. Since X is connected, the complement of the union of the closed sets $\mathrm{supp}(f)$ and $\mathrm{supp}(g)$ cannot be empty, and contains some point x_0. Now ψ restricts to a cocycle

$\psi_{X-\{x_0\}}$ on $C^\infty_c(X-\{x_0\})$, which is diagonal because $X-\{x_0\}$ is noncompact. In particular, $\psi(f,g)=\psi_{X-\{x_0\}}(f,g)=0$, so that ψ is diagonal. $\square$

In the remainder of this section, we will focus mainly on the second Lie algebra cohomology of the Poisson Lie algebra. The following proposition shows that this suffices in order to determine $H^2(\mathrm{ham}(X),\mathbb{R})$.

Proposition 4.4. *For any connected symplectic manifold X, the canonical projection $f\mapsto X_f$ induces a surjection $H^2(\mathrm{ham}(X),\mathbb{R})\to H^2(C^\infty(X),\mathbb{R})$. If X is compact, this is an isomorphism*

$$H^2(\mathrm{ham}(X),\mathbb{R})\simeq H^2(C^\infty(X),\mathbb{R})\,.$$

If X is noncompact, then its kernel is 1-dimensional, spanned by the Kostant-Souriau class $[\psi_{KS}]$ of equation (34). Every splitting of the exact sequence

$$\mathbb{R}[\psi_{KS}]\to H^2(\mathrm{ham}(X),R)\to H^2(C^\infty(X),\mathbb{R})$$

of vector spaces yields an isomorphism

$$H^2(\mathrm{ham}(X),R)\simeq H^2(C^\infty(X),\mathbb{R})\oplus\mathbb{R}[\psi_{KS}].$$

Proof. Every cocycle $\psi\colon C^\infty(X)\times C^\infty(X)\to\mathbb{R}$ vanishes on $C^\infty(X)\times\mathbb{R}\mathbf{1}$. Indeed, write $f=\sum_{i=1}^m\{g_i,h_i\}+c\mathbf{1}$. (We have $C^\infty(X)=[C^\infty(X),C^\infty(X)]$ if X is noncompact and $C^\infty(X)=[C^\infty(X),C^\infty(X)]\oplus\mathbb{R}\mathbf{1}$ if X is compact.) Then

$$\psi(f,\mathbf{1})=\sum_{i=1}^m\psi(\{g_i,h_i\},\mathbf{1})=-\psi(\{h_i,\mathbf{1}\},g_i)-\psi(\{\mathbf{1},g_i\},h_i)=0\,. \tag{39}$$

Since $\mathrm{ham}(X)=C^\infty(X)/\mathbb{R}\mathbf{1}$, 2-cocycles on $\mathrm{ham}(X)$ correspond precisely to 2-cocycles on $C^\infty(X)$. A 1-cochain χ on $C^\infty(X)$ corresponds to a 1-cochain on $\mathrm{ham}(X)$ if and only if $\chi(\mathbf{1})=0$. Writing $\chi=\chi_0+\chi(\mathbf{1})\mathrm{ev}_x$ with $\chi_0:=\chi-\chi(\mathbf{1})\mathrm{ev}_x$ a 1-cochain on $C^\infty(X)/\mathbb{R}\mathbf{1}\simeq\mathrm{ham}(X)$, we see that every 2-coboundary $\mathtt{d}\chi$ on $C^\infty(X)$ is the sum of a coboundary $\mathtt{d}\chi_0$ in $\mathrm{ham}(X)$ and a multiple of the Kostant-Souriau cocycle $\mathtt{d}\mathrm{ev}_x=\psi_{\mathrm{KS}}$. The latter is cohomologous to zero in $\mathrm{ham}(X)$ if and only if X is compact by Corrolary 3.6, so the result follows. $\square$

4.2 Cocycles and coadjoint derivations

If $\mathfrak{g}$ is a locally convex topological Lie algebra, we denote by $\mathfrak{g}'$ the continuous coadjoint representation of $\mathfrak{g}$, that is, the continuous dual with the action

$$(\mathrm{ad}^*_f\phi)(g):=\phi(-[f,g])\,.$$

A *(coadjoint) derivation* $D\colon\mathfrak{g}\to\mathfrak{g}'$ is a linear map satisfying

$$D([f,g])=\mathrm{ad}^*_fD(g)-\mathrm{ad}^*_gD(f)\,.$$

A derivation $\mathfrak{g}\to\mathfrak{g}'$ is *skew symmetric* if $D(f)(g)+D(g)(f)=0$ for all $f,g\in\mathfrak{g}$, and *inner* if it is of the form $D(f)=\mathrm{ad}^*_fH$ for some $H\in\mathfrak{g}'$. It will be convenient to formulate the second Lie algebra cohomology in terms of skew symmetric derivations.

Proposition 4.5. *If $\mathfrak{g}$ is a locally convex Lie algebra, then every continuous 2-cocycle $\psi\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{R}$ induces a skew symmetric derivation*

$$D_\psi(f)\colon g \mapsto \psi(f,g)\,.$$

It is a coboundary (that is, $\psi = \mathtt{d}H$ for some $H\in\mathfrak{g}'$) if and only if D_ψ is inner (that is, $D_\psi(f) = \mathrm{ad}^*_f H$ for some $H\in\mathfrak{g}'$). Conversely, every skew symmetric derivation D induces a separately continuous 2-cocycle

$$\psi_D(f,g) := D(f)(g) \tag{40}$$

on $\mathfrak{g}$. If $\mathfrak{g}$ is either a Fréchet Lie algebra or if $\mathfrak{g} = C^\infty_c(X)$, then ψ_D is automatically jointly continuous. In this case, then, $H^2(\mathfrak{g},\mathbb{R})$ is the space of skew symmetric derivations modulo the inner ones.

Proof. Since ψ is continuous, so is $D_\psi(f)\colon\mathfrak{g}\to\mathbb{R}$. Skew symmetry follows from $\psi(f,g) = -\psi(g,f)$, and the cocycle condition

$$\psi(\{f,g\},h) = \psi(g,-\{f,h\}) - \psi(f,-\{g,h\})$$

is precisely the requirement that D_ψ be a derivation. Conversely, every skew symmetric derivation $D\colon\mathfrak{g}\to\mathfrak{g}'$ defines a 2-cocycle ψ_D by (40). The derivation D_ψ gives back the cocycle ψ, and $\psi = \mathtt{d}H$ if and only if D_ψ is inner, $D_\psi(f) = \mathrm{ad}^*_f H$. Since ψ_D is skew symmetric and continuous in the right argument, it is a *separately* continuous bilinear map on $\mathfrak{g}$, and every inner derivation defines a jointly continuous coboundary.

If $\mathfrak{g}$ is a Fréchet Lie algebra (such as $\mathfrak{g} = C^\infty(X)$ or $\mathfrak{g} = \mathrm{ham}(X)$), then ψ_D is jointly continuous by [Rud91, Thm. 2.17]. If $\mathfrak{g} = C^\infty_c(X)$ for a symplectic manifold X, then by skew symmetry, $D\colon C^\infty_c(X)\to C^\infty_c(X)'$ is continuous for the weak topology, hence given by a distribution on $X\times X$ by the Schwartz Kernel Theorem [Die78, §23.9.2]. In particular, it is jointly continuous. ☐

Remark 4.6. The first Lie algebra cohomology $H^1(\mathfrak{g},\mathfrak{g}')$ of $\mathfrak{g}$ with values in $\mathfrak{g}'$ is defined as the vector space of derivations $D\colon\mathfrak{g}\to\mathfrak{g}'$ modulo the inner derivations. Note that a 2-cocycle ψ is a coboundary, $\psi = \mathtt{d}H$ for some $H\in\mathfrak{g}'$, if and only if D_ψ is inner, $D_\psi(f) = \mathrm{ad}^*_f H$. From Proposition 4.5, we thus obtain a natural map

$$H^2(\mathfrak{g},\mathbb{R}) \to H^1(\mathfrak{g},\mathfrak{g}')\colon\quad [\psi]\mapsto[D_\psi]\,. \tag{41}$$

Remark 4.7. For a weakly hamiltonian action $A\colon G\to\mathrm{Diff}(X)$ (cf. Remark 3.5), the infinitesimal action $a\colon\mathfrak{g}\to\mathrm{ham}(X)$ is G-equivariant. Since also the projection $\pi\colon C^\infty(X)\to\mathrm{ham}(X)$ is G-equivariant, commutativity of the diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, hook] & C^\infty(X) \arrow[r, two heads, "\pi"] & \mathrm{ham}(X) \\
& & \mathfrak{g} \arrow[ul, "J"] \arrow[u, "a"']
\end{tikzcd}$$

shows that $\theta_J(g) := A(g)^*\circ J - J\circ\mathrm{Ad}_{g^{-1}}$ satisfies $\pi\circ\theta_J(g) = 0$, hence takes values in $\mathbb{R}$. The map $\theta_J\colon G\to\mathfrak{g}'$ is a 1-cocycle, $\theta_J(gh) = \theta_J(g) + \mathrm{Ad}^*_g\theta_J(h)$, whose class $[\theta_J]\in H^1(G,\mathfrak{g}')$ does not depend on the choice of J. It vanishes if and only if the action has a G-equivariant comomentum map [Sou70, p. 108–115]. The derived class $[D\theta_J]\in H^1(\mathfrak{g},\mathfrak{g}')$ in Lie algebra cohomology is the image of the class $[\psi_J]\in H^2(\mathfrak{g},\mathbb{R})$ of Remark 3.5 under the map $H^2(\mathfrak{g},\mathbb{R})\to H^1(\mathfrak{g},\mathfrak{g}')$ of Remark 4.6.

In the following lemma, we use Peetre's Theorem to show that for $\mathfrak{g} = C_c^\infty(X)$, every skew-symmetric derivation is a differential operator. We choose a locally finite cover of X by open, relatively compact neighbourhoods U_i with Darboux coordinates x^μ with $\mu \in \{1, \ldots, 2n\}$. Using the summation convention, the Poisson bracket in Darboux coordinates becomes

$$\{f, g\} = \omega^{\sigma\tau} \partial_\sigma f \partial_\tau g \,,$$

where $\omega_{\sigma\tau}$ denote the elements of the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

and $\omega^{\sigma\tau} = -\omega_{\sigma\tau}$ denote the elements of its inverse $J^{-1} = -J$.

We write $\partial_{\vec{\mu}}$ for $\partial_{\mu_1} \ldots \partial_{\mu_N}$ if $\vec{\mu}$ is the multi-index $\vec{\mu} = (\mu_1, \ldots, \mu_N)$ in $\{1, \ldots, 2n\}^N$. Note that if $\vec{\mu}'$ is a permutation of $\vec{\mu}$, i.e. $\mu'_i = \mu_{\sigma(i)}$ with $\sigma \in S_N$, then $\partial_{\vec{\mu}'} = \partial_{\vec{\mu}}$. We write $\vec{\mu}\vec{\nu}$ for the concatenation in $\{1, \ldots, 2n\}^{N+M}$ of $\vec{\mu} \in \{1, \ldots, 2n\}^N$ and $\vec{\nu} \in \{1, \ldots, 2n\}^M$, and we write $|\vec{\mu}|$ for the length N of $\vec{\mu} \in \{1, \ldots, 2n\}^N$. There is one multi-index $\vec{\mu} = *$ for $N = 0$.

Lemma 4.8. *Every skew symmetric derivation $D\colon C_c^\infty(X) \to C_c^\infty(X)'$ is support decreasing, hence determined by its restrictions $D_{U_i}\colon C_c^\infty(U_i) \to C_c^\infty(U_i)'$. In local coordinates, it is a differential operator*

$$D_{U_i}(f) = \sum_{\vec{\mu}} (\partial_{\vec{\mu}} f) S^{\vec{\mu}} \tag{42}$$

of locally finite order, with distributions $S^{\vec{\mu}} \in C_c^\infty(U_i)'$ that are uniquely determined by D and the requirement that they are invariant under permutations of $\vec{\mu}$.

Proof. Since ψ_D is diagonal (Prop. 4.3), D is support decreasing, $\mathrm{supp}(D(f)) \subseteq \mathrm{supp}(f)$. The fact that ψ_D is jointly continuous (Prop. 4.5) implies that the derivation $D\colon C_c^\infty(X) \to C_c^\infty(X)'$ is continuous in the topology on $C_c^\infty(X)'$ obtained from the seminorms $p^*_{K,\vec{\mu}}$ dual to the seminorms $p_{K,\vec{\mu}}$ that define the Fréchet topology on $C_K^\infty(X)$. If we choose a partition of unity λ_i subordinate to the cover U_i, then the expression for D_{U_i} follows from Peetre's theorem [Pee60, Thm. 1] applied to the support decreasing maps $\lambda_j D(\lambda_i \,\cdot\,)\colon C_c^\infty(U_i) \to C_c^\infty(U_j)'$, which have empty set of discontinuities. □

If we consider $C^\infty(X)$ instead of $C_c^\infty(X)$, then the only thing that changes is that the relevant cochains must have compact support. The following lemma gives the relation between the two situations, and essentially allows us to restrict attention to $C_c^\infty(X)$.

Lemma 4.9. *The cocycle ψ_D extends continuously to $C^\infty(X)$ if and only if D is of compact support. If $\psi_D = \mathtt{d}\chi$ for $\chi \in C_c^\infty(X)'$, then χ extends to $C^\infty(X)$. In particular, the map $\iota^*\colon H^2(C^\infty(X), \mathbb{R}) \to H^2(C_c^\infty(X), \mathbb{R})$ induced by the inclusion $\iota\colon C_c^\infty(X) \to C^\infty(X)$, is injective.*

Proof. Suppose that ψ_D extends to a continuous cocycle on $C^\infty(X)$. If $\mathrm{supp}(D)$ were not compact, there would exist a countably infinite, locally finite set of points $x_i \in \mathrm{supp}(\psi_D)$. We would find disjoint open sets $V_i \ni x_i$ and functions

f_i, g_i, supported in V_i, such that $D(f_i)(g_i) = 1$. Since the sum $\sum_{i=1}^{\infty}(f_i, g_i)$ converges in $C^{\infty}(X) \times C^{\infty}(X)$, so would $\sum_{i=1}^{\infty} \psi_D(f_i, g_i) = \sum_{i=1}^{\infty} 1$, which is absurd. We conclude that if ψ_D extends continuously to $C^{\infty}(X)$, then $\mathrm{supp}(\psi_D)$ is compact. The converse implication is clear.

It remains to show that if ψ_D extends to $C^{\infty}(X)$ and $\psi_D = \mathtt{d}\chi$ on $C^{\infty}_c(X)$, then $\mathrm{supp}(\chi)$ is compact, so that χ extends to $C^{\infty}(X)$. Suppose $\mathrm{supp}(\chi)$ were not compact. Then again, we find disjoint open sets $V_i \subseteq X$ and $F_i \in C^{\infty}_c(V_i)$ with $\chi(F_i) = 1$. Using Remark 3.4 to write $F_i = -\sum_{k=1}^{2n(2n+1)}\{f_i^k, g_i^k\}$ with $f_i^k, g_i^k \in C^{\infty}_c(V_i)$, we find $\sum_{k=1}^{2n(2n+1)} \psi(f_i^k, g_i^k) = 1$. Since $\sum_{i=1}^{\infty}(f_i^k, g_i^k)$ converges in $C^{\infty}(X) \times C^{\infty}(X)$, we see that $\sum_{i=1}^{\infty}\sum_{k=1}^{2n(2n+1)} \psi(f_i^k, g_i^k) = \sum_{i=1}^{\infty} 1$ converges, which is absurd. We conclude that $\mathrm{supp}(\chi)$ is compact. $\square$

Sanctioned by Lemma 4.9, we restrict attention to $C^{\infty}_c(X)$. By looking at the cocycle equation in local coordinates, we now prove that the differential operator D corresponding to a continuous 2-cocycle ψ_D can have degree at most 1. In Lemma 4.11, we will use this to derive explicit local expressions for the cocycles.

Lemma 4.10. *If a differential operator $D \colon C^{\infty}_c(X) \to C^{\infty}_c(X)'$ is a derivation, then it is of order at most 1.*

Proof. We express the condition

$$D(\{f, g\}) - \mathrm{ad}^*_f D(g) + \mathrm{ad}^*_g D(f) = 0 \tag{43}$$

that D be a derivation in local Darboux coordinates. Using the summation convention and writing $f_{\vec{\mu}}$ for $\partial_{\vec{\mu}} f$, we have $D_{U_i}(f) = f_{\vec{\mu}} S^{\vec{\mu}}$. The formula

$$\mathrm{ad}^*_f \phi = \omega^{\sigma\tau} f_{\sigma} \phi_{\tau}$$

follows from $\mathrm{ad}^*_f \phi(g) = \phi(-\omega^{\sigma\tau} f_{\sigma} g_{\tau}) = \omega^{\sigma\tau}\partial_{\tau}(f_{\sigma}\phi)(g)$ and the antisymmetry of ω, which implies $\omega^{\sigma\tau}(f_{\sigma\tau}) = 0$. Writing $S^{\vec{\mu}}_{,\tau}$ for $\partial_{\tau} S^{\vec{\mu}}$, and using that $\omega^{\sigma\tau}$ is constant in Darboux coordinates, we find the local expression for equation (43)

$$\omega^{\sigma\tau}\Big(\big(\sum_{\vec{\alpha}\cup\vec{\beta}=\vec{\mu}} f_{\sigma\vec{\alpha}} g_{\tau\vec{\beta}}\big) S^{\vec{\mu}} - f_{\sigma} g_{\vec{\mu}} S^{\vec{\mu}}_{,\tau} - f_{\sigma} g_{\tau\vec{\mu}} S^{\vec{\mu}} + g_{\sigma} f_{\vec{\mu}} S^{\vec{\mu}}_{,\tau} + g_{\sigma} f_{\tau\vec{\mu}} S^{\vec{\mu}}\Big) = 0\,, \tag{44}$$

where $\sum_{\vec{\alpha}\cup\vec{\beta}=\vec{\mu}}$ denotes the sum over the 2^N decompositions of $\vec{\mu} = (\mu_1, \ldots \mu_N)$ into $\vec{\alpha} = (\mu_{i_1}, \ldots, \mu_{i_K})$ and $\vec{\beta} = (\mu_{j_1}, \ldots, \mu_{j_{N-K}})$, where $1 \leq i_1 < \ldots, i_K \leq N$ and $1 \leq j_1 < \ldots < j_{N-K} \leq N$ complement each other to $1 < \ldots < N$. We can write (44) as

$$f_{\vec{\mu}} g_{\vec{\nu}} Q^{\vec{\mu};\vec{\nu}} = 0 \tag{45}$$

for the distributions

$$Q^{\vec{\mu};\vec{\nu}} := \tfrac{(|\vec{\mu}|+|\vec{\nu}|-2)!}{|\vec{\mu}|!|\vec{\nu}|!} \sum_{\sigma\cup\vec{\alpha}=\vec{\mu}} \sum_{\tau\cup\vec{\beta}=\vec{\nu}} \omega^{\sigma\tau} S^{\vec{\alpha}\vec{\beta}} + \delta_{|\vec{\nu}|,1} Y^{\nu;\vec{\mu}} - \delta_{|\vec{\mu}|,1} Y^{\mu;\vec{\nu}} \tag{46}$$

where the first term on the r.h.s. is zero for $|\vec{\mu}| = 0$ or $|\vec{\nu}| = 0$, and where

$$Y^{\nu;\vec{\mu}} := \omega^{\nu\tau} S^{\vec{\mu}}_{,\tau} + \tfrac{1}{|\vec{\alpha}|+1} \sum_{\tau\cup\vec{\alpha}=\vec{\mu}} \omega^{\nu\tau} S^{\vec{\alpha}}\,. \tag{47}$$

Since the distributions $Q^{\vec{\mu};\vec{\nu}}$ are manifestly independent of f and g, and symmetric under permutation of the $\vec{\mu}$ and $\vec{\nu}$ indices separately, the fact that (45) holds for arbitrary f and g implies

$$Q^{\vec{\mu};\vec{\nu}} = 0\,. \tag{48}$$

In order to show that D is a differential operator of degree at most 1, we will show that, for any multi-index $\vec{\gamma}$ of length ≥ 2, the distribution $S^{\vec{\gamma}}$ is identically zero.

There exists a non-trivial decomposition $\vec{\alpha} \cup \vec{\beta} = \vec{\gamma}$ with $|\vec{\alpha}| = r \geq 1$ and $|\vec{\beta}| = s \geq 1$, since $|\vec{\gamma}| \geq 2$. We know that $Q^{\sigma\cup\vec{\alpha};\tau\cup\vec{\beta}} = 0$ for all $\sigma, \tau \in \{1, \ldots, 2n\}$. Moreover, for these distributions, the contribution of Y in the expression (46) is zero, so, denoting by K the constant $\frac{(r+s)!}{(r+1)!(s+1)!}$, we have

$$Q^{\sigma\cup\vec{\alpha};\tau\cup\vec{\beta}} = K \sum_{\sigma'\cup\vec{\alpha}'=\sigma\cup\vec{\alpha}} \; \sum_{\tau'\cup\vec{\beta}'=\tau\cup\vec{\beta}} \omega^{\sigma'\tau'} S^{\vec{\alpha}'\vec{\beta}'}. \tag{49}$$

We contract $Q^{\sigma\cup\vec{\alpha};\tau\cup\vec{\beta}}$ with $\omega_{\tau\sigma} = -\omega^{\tau\sigma}$, the inverse of ω satisfying $\omega_{\tau\sigma}\omega^{\sigma\gamma} = \delta^{\gamma}_{\tau}$. The result is

$$\omega_{\tau\sigma} Q^{\sigma\cup\vec{\alpha};\tau\cup\vec{\beta}} = (2n + r + s) K S^{\vec{\alpha}\vec{\beta}}. \tag{50}$$

Indeed, there are 4 types of terms on the r.h.s. of (49) when contracted with $\omega_{\sigma\tau}$, according to whether σ' is equal to σ or not, and whether τ' is equal to τ or not.

If $\sigma \neq \sigma'$ and $\tau \neq \tau'$, then σ is in $\vec{\alpha}'$ and τ is in $\vec{\beta}'$, so the term is symmetric under $\sigma \leftrightarrow \tau$. It therefore contracts to zero with the antisymmetric tensor $\omega_{\tau\sigma}$.

If $\sigma = \sigma'$ and $\tau = \tau'$, then because $\omega_{\tau\sigma}\omega^{\sigma\tau} = 2n$, contraction yields the term $2nKS^{\vec{\alpha}\vec{\beta}}$. There is precisely one such term.

If $\sigma = \sigma'$ and $\tau \neq \tau'$, say $\tau' = \beta_p$, then $\omega_{\tau\sigma}\omega^{\sigma\beta_p} = \delta^{\beta_p}_{\tau}$. Together with the symmetry of $S^{\vec{\mu}}$ under permutations of $\vec{\mu}$, this implies that

$$\omega_{\tau\sigma}\omega^{\sigma\beta_p} S^{\vec{\alpha}\tau\beta_1\ldots\hat{\beta}_p\ldots\beta_s} = S^{\vec{\alpha}\vec{\beta}}\,.$$

Since there are s such terms, their contribution is $sKS^{\vec{\alpha}\vec{\beta}}$. By a similar reasoning, the terms with $\sigma \neq \sigma'$ and $\tau = \tau'$ have a contribution of $rKS^{\vec{\alpha}\vec{\beta}}$.

The equality (50), combined with (48), now ensures that $S^{\vec{\gamma}} = S^{\vec{\alpha}\vec{\beta}} = 0$ for $\vec{\gamma}$ of length ≥ 2, so that D is a differential operator of order at most 1. □

We continue this line of reasoning to obtain the following more explicit expression for these derivations.

Lemma 4.11. *If a differential operator $D\colon C^\infty_c(X) \to C^\infty_c(X)'$ is a derivation, then in local Darboux coordinates, it is given by a first order differential operator*

$$D_{U_i}(f) = S^\mu \partial_\mu f + cf$$

for $c \in \mathbb{R}$ and distributions S^μ with

$$\omega^{\nu\tau}\partial_\tau S^\mu - \omega^{\mu\tau}\partial_\tau S^\nu = c\omega^{\mu\nu}\,. \tag{51}$$

The derivation D is antisymmetric if and only if $c = 0$.

Remark 4.12. Contracting both sides of (51) with $\omega_{\mu\nu}$, one obtains

$$\partial_\mu S^\mu = -nc\,.$$

In particular, $\partial_\mu S^\mu = 0$ if D is antisymmetric.

Proof. The requirement that D be a derivation is equivalent to eqn. (48) for the $Q^{\vec{\mu};\vec{\nu}}$ of eqn. (46). Using the antisymmetry of ω, we see that

$$Q^{\mu_1\mu_2;\nu_1} = \frac{1}{2}\big(\omega^{\mu_1\nu_1}S^{\mu_2} + \omega^{\mu_2\nu_1}S^{\mu_1} + \omega^{\nu_1\mu_1}S^{\mu_2} + \omega^{\nu_1\mu_2}S^{\mu_2}\big)$$

vanishes identically, as does $Q^{\mu_1;\nu_1\nu_2}$. For $Q^{\mu;\nu}$ and $Q^{*;\mu} = -Q^{\mu;*}$, eqn. (46) yields the nontrivial equations

$$Q^{\mu;\nu} = \omega^{\nu\tau}\partial_\tau S^\mu - \omega^{\mu\tau}\partial_\tau S^\nu - \omega^{\mu\nu}S^* = 0 \tag{52}$$

and

$$Q^{*;\mu} = \omega^{\mu\tau}\partial_\tau S^* = 0\,. \tag{53}$$

Because ω is invertible, the latter equation implies that S^* is a constant, $S^* = c$. (A distribution satisfies $\partial_\mu\phi = 0$ for all μ if and only if it is constant). Contracting equation (52) with $\omega_{\mu\nu}$ and using antisymmetry of ω, together with $\omega_{\mu\nu}\omega^{\nu\sigma} = \delta^\sigma_\mu$ and $\delta^\mu_\mu = \mathrm{tr}(\mathbf{1}) = 2n$, we obtain $2\partial_\mu S^\mu + 2nS^* = 0$, so that the divergence $\partial_\mu S^\mu = -nc$ is constant. If D is antisymmetric, the equation

$$\begin{aligned} D(f)(g) + D(g)(f) &= (S^\mu(f_\mu g) + S^*(fg)) + (S^\mu(fg_\mu) + S^*(fg)) \\ &= S^\mu(\partial_\mu(fg)) + 2S^*(fg) \\ &= (2S^* - \partial_\mu S^\mu)(fg) = 0 \end{aligned}$$

for all f, g yields $S^* = \frac{1}{2}\partial_\mu S^\mu$, and because we already had $S^* = -\frac{1}{n}\partial_\mu S^\mu$, we get $S^* = c = 0$. □

4.3 Second cohomology of the Poisson Lie algebra

Following Roger [Rog95, §9], we now define maps

$$H^1_{\mathrm{dR}}(X) \to H^2(C^\infty_c(X), \mathbb{R}) \tag{54}$$

and

$$H^1_{\mathrm{dR},c}(X) \to H^2(C^\infty(X), \mathbb{R})\,, \tag{55}$$

from the (compactly supported) de Rham cohomology of X into the continuous Lie algebra cohomology of the Poisson Lie algebras $C^\infty_c(X)$ and $C^\infty(X)$. We then use the preceding results to show that these maps are isomorphisms.

Proposition 4.13. *If $\alpha \in \Omega^1(X)$ is closed, then*

$$\psi_\alpha(f,g) := \int_X f(i_{X_g}\alpha)\,\omega^n/n! \tag{56}$$

defines a 2-cocycle on $C^\infty_c(X)$, which is a coboundary if α is exact, $\alpha = dh$ with $h \in \Omega^0(X)$. If α is compactly supported, then ψ_α extends to a cocycle on $C^\infty(X)$, which is a coboundary if $\alpha = dh$ with $h \in \Omega^0_c(X)$. In particular, the correspondence

$$[\alpha] \mapsto [\psi_\alpha]\,. \tag{57}$$

yields well defined maps (54) *and* (55).

Proof. If α is closed, then the vector field S defined by $i_S\omega = -\alpha$ is symplectic, i.e. $L_S\omega = 0$. In the following, we either have $f, g \in C_c^\infty(X)$ and $S \in \mathrm{sp}(X)$ or $f, g \in C^\infty(X)$ and $S \in \mathrm{sp}_c(X)$. Either way, we have $L_S\{f,g\} = \{L_Sf, g\} + \{f, L_Sg\}$, so L_S defines a derivation into $C_c^\infty(X)$. If we define a linear functional $\langle\,\cdot\,\rangle$ on the symmetric tensor product $C_c^\infty(X) \vee C^\infty(X)$ by

$$\langle f \vee g\rangle := \int_X fg\omega^n/n!\,,$$

then $\langle\,\cdot\,\rangle$ is L_S-invariant in the sense that

$$\langle L_Sf \vee g\rangle + \langle f \vee L_Sg\rangle = 0\,, \tag{58}$$

because $\int_X (L_Sf)g\omega^n + \int_X f(L_Sg)\omega^n = \int_X L_S(fg\omega^n) = 0$. Since $i_{X_f}\alpha = -L_Sf$, we can write

$$\psi_\alpha(f,g) = -\langle f \vee L_Sg\rangle\,. \tag{59}$$

Equation (58) then ensures that ψ_α is skew symmetric, $\psi_\alpha(f,g) + \psi_\alpha(g,f) = 0$. The cocycle identity follows from

$$\begin{aligned}
\langle f \vee L_S\{g,h\}\rangle + \langle g \vee L_S\{h,f\}\rangle + \langle h \vee L_S\{f,g\}\rangle &= \\
\langle f \vee (\{L_Sg,h\} + \{g, L_Sh\})\rangle - \langle L_Sg \vee \{h,f\}\rangle - \langle L_Sh \vee \{f,g\}\rangle &= \\
\langle f \vee (\{L_Sg,h\} + \{g, L_Sh\})\rangle - \langle \{L_Sg,h\} \vee f\rangle - \langle \{g, L_Sh\} \vee f\rangle &= 0\,,
\end{aligned}$$

where in the last step, we used that (58) with $S = X_h$ implies $\langle\{h,f\} \vee g\rangle + \langle f, \{h,g\}\rangle = 0$. If α is exact, say $\alpha = dh$ for $h \in C^\infty(X)$ not necessarily compactly supported, then $S = X_h$ is a hamiltonian vector field, and

$$\psi_{dh}(f,g) = -\langle f \vee L_{X_h}g\rangle = \langle f \vee L_{X_g}h\rangle = \langle\{f,g\} \vee h\rangle \tag{60}$$

shows that $\psi_{dh} = -\mathtt{d}\chi_h$ for the 1-cochain $\chi_h(f) := \langle f \vee h\rangle$. $\square$

We can define a similar map for the Lie algebra $\mathrm{ham}(X)$, but here we have to be slightly careful; every cocycle ψ_α on $C^\infty(X)$ yields a cocycle on $\mathrm{ham}(X)$, but even if ψ_α is a coboundary on $C^\infty(X)$, it need not be trivial on $\mathrm{ham}(X)$. It turns out that if $\alpha = dh$ with $h \in \Omega_c^0(X)$, then $[\psi_\alpha] = 0$ in $H^2(C^\infty(X), \mathbb{R})$, but $[\psi_\alpha] = \langle h\rangle[\psi_{KS}]$ in $H^2(\mathrm{ham}(X), \mathbb{R})$, where

$$\langle h\rangle := \lambda(h) = \int_X h\omega^n/n!\,. \tag{61}$$

Proposition 4.14. *Let X be a symplectic manifold. Assigning to the closed 1-form $\alpha \in \Omega_c^1(X)$ the cocycle*

$$\psi_\alpha(X_f, X_g) = \int_X f\alpha(X_g)\omega^n/n! \tag{62}$$

yields a well defined linear map $Z_c^1(X)/d\Omega_{c,0}^0(X) \to H^2(\mathrm{ham}(X), \mathbb{R})$, where

$$Z_c^1(X) := \mathrm{Ker}(d\colon \Omega_c^1(X) \to \Omega_c^2(X))$$

and

$$\Omega_{c,0}^0(X) := \Big\{h \in \Omega_c^0(X)\,;\, \langle h\rangle = 0\Big\}\,.$$

In particular, we have a map $H_{\mathrm{dR}}^1(X) \to H^2(\mathrm{ham}(X), \mathbb{R})$ if X is compact.

Proof. Equation (59) shows that $\psi_\alpha(f,g)$ vanishes if $g=1$, so that ψ_α defines a cocycle on $\mathrm{ham}(X)$. If $\alpha = dh$, then we choose a point $x \in X$, and define the 1-cochain $\chi_h(X_f) := \int_X (f_x - f)h\omega^n/n!$ on $\mathrm{ham}(X)$. Combining

$$\mathtt{d}\chi_h(X_f, X_g) = \int_X \{f,g\}h\omega^n/n! - \{f,g\}_x \int_X h\omega^n/n!$$

with equations (60), (34), and (61), we find $[\psi_{dh}] = \langle h\rangle[\psi_{KS}]$. ☐

From Prop. 4.5 we know that every continuous 2-cocycle of the compactly supported Poisson Lie algebra $C^\infty_c(X)$ comes from a skew symmetric derivation. These derivations are explicitly described in Lemmas 4.8, 4.10 and 4.11. We see that the restriction of the cocycle ψ to a Darboux coordinate patch U_i can be written as

$$\psi(f,g) = S^\mu_{U_i}(g\partial_\mu f) \tag{63}$$

for distributions $S^\mu_{U_i}$ on U_i satisfying

$$\omega^{\nu\tau}\partial_\tau S^\mu_{U_i} - \omega^{\mu\tau}\partial_\tau S^\nu_{U_i} = 0. \tag{64}$$

We now use this explicit description of 2-cocycles to show that the map (54) is an isomorphism.

Theorem 4.15. *For any symplectic manifold X, the map $[\alpha] \mapsto [\psi_\alpha]$ described in Prop. 4.13 is an isomorphism $H^1_{\mathrm{dR}}(X) \simeq H^2(C^\infty_c(X), \mathbb{R})$.*

Proof. Let ψ be a continuous Lie algebra 2-cocycle on the Poisson Lie algebra $C^\infty_c(X)$. The restriction of ψ to a Darboux coordinate patch U_i is given by (63) for distributions $S^\mu_{U_i}$ on U_i satisfying (64). We define the TX-valued distribution $S_{U_i} \in \Omega^1_c(U_i)'$ by $S_{U_i}(\alpha) = S^\mu_{U_i}(\alpha_\mu)$, where $\alpha = \alpha_\mu dx^\mu$ on U_i. Then S_{U_i} agrees with S_{U_j} on the overlap $U_i \cap U_j$ because there, both are determined uniquely by the expression $S_{U_i}(gdf) = \psi(f,g) = S_{U_j}(gdf)$ for $f,g \in C^\infty_c(U_i \cap U_j)$. The TX-valued distributions S_{U_i} therefore splice together to a TX-valued distribution $S \in \Omega^1_c(X)'$ such that[1]

$$\psi(f,g) = S(gdf)\,. \tag{65}$$

The requirement $\omega^{\nu\tau}\partial_\tau S^\mu_{U_i} - \omega^{\mu\tau}\partial_\tau S^\nu_{U_i} = 0$ for the local distributions translates to $S \circ \delta = 0$, for $S \in \Omega^1_c(X)'$, with $\delta\colon \Omega^2_c(X) \to \Omega^1_c(X)$ the canonical homology operator of equation (19). Indeed, using equation (22), one calculates that, in Darboux coordinates, we have

$$\delta F_{\sigma\tau}dx^\sigma \wedge dx^\tau = -\omega^{\nu\tau}\partial_\tau F_{\nu\mu}dx^\mu + \omega^{\mu\tau}\partial_\tau F_{\nu\mu}dx^\nu\,,$$

so that for $F \in \Omega^2_c(U_i)$, we have $S(\delta F) = \omega^{\nu\tau}\partial_\tau S^\mu_{U_i}(F_{\nu\mu}) - \omega^{\mu\tau}\partial_\tau S^\nu_{U_i}(F_{\nu\mu}) = 0$. We conclude that continuous Lie algebra 2-cocycles ψ on $C^\infty_c(X)$ correspond bijectively to continuous linear maps $S\colon \Omega^1_c(X)/\delta(\Omega^2_c(X)) \to \mathbb{R}$. In particular, every continuous 2-cocycle ψ yields an element of $H^{\mathrm{can}}_{c,1}(X)^*$ by restricting S to $\mathrm{Ker}(\delta)/\delta(\Omega^2_c(X)) = H^{\mathrm{can}}_{c,1}(X)$.

Note that, if $\psi = \mathtt{d}H$ is the boundary of $H \in C^\infty_c(X)'$ in Lie algebra cohomology, then, using equation (22), we see that $\psi(f,g) = H(-\{f,g\})$ implies $S(gdf) = H(\delta(gdf))$. It follows that ψ is exact if and only if $S = H \circ \delta$ for some

[1]Note that this differs from the convention in (12) by a minus sign.

$H \in C_c^\infty(X)'$. In particular, cohomologous cocycles induce the same element of $H_{c,1}^{\mathrm{can}}(X)^*$. We thus obtain a linear map

$$[\psi] \in H^2(C_c^\infty(X), \mathbb{R}) \mapsto S \in H_{c,1}^{\mathrm{can}}(X)^*. \tag{66}$$

We prove that this map is injective. If $[\psi] \in H^2(C_c^\infty(X), \mathbb{R})$ maps to zero, then $S\colon \Omega_c^1(X) \to \mathbb{R}$ vanishes on $\mathrm{Ker}(\delta)$, and defines a continuous linear functional $S\colon \Omega_c^1(X)/\mathrm{Ker}(\delta) \to \mathbb{R}$. Since $\delta\colon \Omega_c^1(X) \to \mathrm{Im}(\delta) = C_{c,0}^\infty(X)$ is a continuous, surjective map of LF-spaces, it is open by [DS49, Thm. 1], so that

$$\delta\colon \Omega_c^1(X)/\mathrm{Ker}(\delta) \to \mathrm{Im}(\delta) = C_{c,0}^\infty(X) \subseteq \Omega_c^0(X)$$

is an isomorphism of topological vector spaces. It follows that S can be written as $S = H_0 \circ \delta$ for a unique continuous functional $H_0\colon \mathrm{Im}(\delta) \to \mathbb{R}$. Since $\mathrm{Im}(\delta) \subseteq \Omega_c^0(X)$ is a complemented inclusion of locally convex spaces (one can choose a compactly supported function f with $\int_{X_x} f\omega^n/n!$ on every connected component X_x of X), H_0 extends to a continuous functional $H\colon \Omega_c^0(X) \to \mathbb{R}$. Since $S = H \circ \delta$, we have $[\psi] = 0$ by (65).

It remains to show that the map (66) is surjective. Recall that $H_{c,1}^{\mathrm{can}}(X)^*$ is isomorphic to $H_{\mathrm{dR},c}^{2n-1}(X)^*$ under the symplectic hodge star operator, which in turn is isomorphic to $H_{\mathrm{dR}}^1(X)$ by Poincaré duality. Proposition 4.13 therefore yields a map $H_{\mathrm{dR}}^1(X) \to H^2(C_c^\infty(X), \mathbb{R})$ in the other direction, and we show that it is a left inverse to (66). Given a class $[\alpha] \in H_{\mathrm{dR}}^1(X)$, the corresponding cocycle

$$\psi_\alpha(f,g) := -\int_X g(i_{X_f}\alpha)\,\omega^n/n!$$

can also be written $\psi_\alpha(f,g) = \int_X \alpha \wedge *gdf$, as

$$\begin{aligned} -g(i_{X_f}\alpha)\omega^n/n! &= -g\alpha \wedge i_{X_f}\omega^n/n! \\ &= \alpha \wedge gdf \wedge \omega^{n-1}/(n-1)! \\ &= \alpha \wedge *gdf\,. \end{aligned}$$

Therefore, we have

$$S(\beta) = \int_X \alpha \wedge *\beta$$

for the TX-valued distribution S associated to ψ_α. Under the symplectic hodge star operator, the induced map $S\colon H_{c,1}^{\mathrm{can}}(X) \to \mathbb{R}$ corresponds with the map $H_{\mathrm{dR},c}^{2n-1}(X) \to \mathbb{R}$ given by $[\beta] \mapsto \int_X \beta\wedge\alpha$. Since this corresponds to $[\alpha] \in H_{\mathrm{dR}}^1(X)$ under Poincaré duality, the composition $H_{\mathrm{dR}}^1(X) \to H^2(C_c^\infty(X), \mathbb{R}) \to H_{\mathrm{dR}}^1(X)$ is the identity, and surjectivity of (66) follows. We conclude that the map $[\alpha] \mapsto [\psi_\alpha]$ is an isomorphism. $\square$

Using Lemma 4.9, we now show that also the map (55) is an isomorphism.

Theorem 4.16. *For any symplectic manifold X, the map $[\alpha] \mapsto [\psi_\alpha]$ described in Prop. 4.13, with*

$$\psi_\alpha(f,g) = \int_X f(i_{X_g}\alpha)\,\omega^n/n!, \tag{67}$$

is an isomorphism $H_{\mathrm{dR},c}^1(X) \simeq H^2(C^\infty(X), \mathbb{R})$.

Proof. This is a straightforward adaptation of the proof of Theorem 4.15. Let ψ be a continuous 2-cocycle on $C^\infty(X)$. Since its restriction to $C^\infty_c(X)$ extends continuously to $C^\infty(X)$, the corresponding TX-valued distribution S is of compact support, $S \in \Omega^1(X)'$. It vanishes on $\delta\Omega^2(X)$, so S defines a continuous linear map $S\colon \Omega^1(X)/\delta\Omega^2(X) \to \mathbb{R}$, hence an element of $H_1^{\mathrm{can}}(X)^*$. If $\psi = H \circ \delta$, then H is compactly supported by Lemma 4.9, so the element of $H_1^{\mathrm{can}}(X)^*$ depends only on the class of $[\psi]$. The map $H^2(C^\infty(X), \mathbb{R}) \to H_1^{\mathrm{can}}(X)^*$ is bijective by an argument similar to the one in Theorem 4.15, with the map (55) of Prop. 4.13 taking the role of left inverse, $\Omega^0(X)$ takes the role of $\Omega^0_c(X)$, and we use $H^{2n-1}_{\mathrm{dR}}(X)$ instead of $H^{2n-1}_{\mathrm{dR},c}(X)$. $\square$

Combining Proposition 4.4 with Theorem 4.16, we obtain the following description of the second Lie algebra cohomology of $\mathrm{ham}(X)$.

Theorem 4.17. *Let X be a connected symplectic manifold. Then the map from Proposition 4.14, assigning to the closed 1-form $\alpha \in \Omega^1_c(X)$ the cocycle*

$$\psi_\alpha(X_f, X_g) = \int_X f\alpha(X_g)\omega^n/n!, \tag{68}$$

is an isomorphism

$$Z^1_c(X)/d\Omega^0_{c,0}(X) \to H^2(\mathrm{ham}(X), \mathbb{R}).$$

In particular, $H^2(\mathrm{ham}(X), \mathbb{R})$ is isomorphic to $H^1_{\mathrm{dR}}(X)$ if X is compact and to $H^1_{\mathrm{dR},c}(X) \oplus \mathbb{R}[\psi_{KS}]$ if X is noncompact.

Note that for noncompact X the isomorphism between $Z^1_c(X)/d\Omega^0_{c,0}(X)$ and $H^1_{\mathrm{dR},c}(X) \oplus \mathbb{R}[\psi_{KS}]$ is not canonical.

5 Universal central extension of $\mathrm{ham}(X)$

Our results on the second Lie algebra cohomology of $\mathrm{ham}(X)$ yield an explicit classification of the continuous central extensions of $\mathrm{ham}(X)$ by $\mathbb{R}$. In this section, we will use this to prove that, for a connected symplectic manifold X with $H^{2n-1}_{\mathrm{dR}}(X)$ finitely generated, the central extension described in the following proposition is the universal central extension of $\mathrm{ham}(X)$. We will then see that the same Lie algebra also yields a universal central extension of $C^\infty(X)$ if X is noncompact.

Proposition 5.1. *Let X be a symplectic manifold. Then $\Omega^1(X)/\delta\Omega^2(X)$ is a Fréchet Lie algebra with Lie bracket*

$$[[\alpha], [\beta]] := [\delta\alpha \cdot d\delta\beta]. \tag{69}$$

If we define $\pi\colon \Omega^1(X)/\delta\Omega^2(X) \to \mathrm{ham}(X)$ by $\pi([\alpha]) := X_{\delta\alpha}$, then

$$\mathfrak{z} \xrightarrow{\iota} \Omega^1(X)/\delta\Omega^2(X) \xrightarrow{\pi} \mathrm{ham}(X) \tag{70}$$

is a continuous central extension of $\mathrm{ham}(X)$ by the center of $\Omega^1(X)/\delta\Omega^2(X)$, which is given by

$$\mathfrak{z} = \mathrm{Ker}(d \circ \delta)/\delta\Omega^2(X). \tag{71}$$

Remark 5.2. If X is compact, then $\mathrm{Ker}(d \circ \delta) = \mathrm{Ker}(\delta)$, since $\delta(\Omega^1(X)) = C^\infty_{c,0}(X)$ by Lemma 2.11. It follows that $\mathfrak{z}$ equals $H_1^{\mathrm{can}}(X)$, which is isomorphic to $H^{2n-1}_{\mathrm{dR}}(X)$ by Prop. 2.4. If X is noncompact, then $\mathfrak{z}$ is slightly bigger than $H^{2n-1}_{dR}(X)$. In fact, we have an exact sequence

$$H^{2n-1}_{\mathrm{dR}}(X) \xrightarrow{*} \mathfrak{z} \xrightarrow{\delta} \mathbb{R}\,,$$

where the extra dimension corresponds to the Kostant-Souriau extension.

Remark 5.3. We obtain a different description of the Lie algebra $\Omega^1(X)/\delta\Omega^2(X)$ if we identify it with $\Omega^{2n-1}(X)/d\Omega^{2n-2}(X)$ using the symplectic hodge star operator. The Lie bracket on $\Omega^{2n-1}(X)/d\Omega^{2n-2}(X)$ takes the form

$$[[\gamma_1], [\gamma_2]] = [f_1 df_2 \wedge \omega^{n-1}/(n-1)!],$$

where the function $f \in C^\infty(X)$ is uniquely defined by the form $\gamma \in \Omega^{2n-1}(X)$ via $f\omega^n/n! = d\gamma$. The Lie algebra homomorphism $\Omega^{2n-1}(X)/d\Omega^{2n-2}(X) \to \mathrm{ham}(X)$ is given by $[\gamma] \mapsto X_f$, yielding the central extension

$$\mathrm{Ker}(\delta \circ d)/d\Omega^{2n-2}(X) \to \Omega^{2n-1}(X)/d\Omega^{2n-2}(X) \to \mathrm{ham}(X) \qquad (72)$$

If X is compact, then $\mathrm{Ker}(\delta \circ d)/d\Omega^{2n-2}(X) = H^{2n-1}_{\mathrm{dR}}(X)$.

Proof. To show that $\Omega^1(X)/\delta\Omega^2(X)$ is Fréchet, it suffices to exhibit $\delta\Omega^2(X)$ as a closed subspace of the Fréchet space $\Omega^1(X)$ (cf. [Rud91, Thm 1.41]). Since $\delta = (-1)^{k+1} * d\, *$ and $*^2 = \mathrm{Id}$, the equality $\alpha = \delta F$ for $F \in \Omega^2(X)$ is equivalent to $*\alpha = d * F$, so that $\alpha \in \Omega^1(X)$ is δ-exact if and only if $\int_X (*\alpha) \wedge \beta = 0$ for all closed $\beta \in \Omega^1_c(X)$. Since this is a closed condition, $\delta\Omega^2(X)$ is closed in $\Omega^1(X)$.

The bracket (69) is continuous because it is given in terms of differential operators. Moreover, π preserves the brackets because by (23), we have

$$\delta[[\alpha], [\beta]] = \delta(\delta\alpha \cdot d\delta\beta) = \{\delta\alpha, \delta\beta\}\,, \qquad (73)$$

so that $X_{\delta[[\alpha],[\beta]]} = [X_{\delta\alpha}, X_{\delta\beta}]$.

To check that (69) defines a Lie bracket, we use the following identity proven in Lemma 2.3:

$$d\Omega^0(X) \subseteq \delta\Omega^{2n-2}(X)\,. \qquad (74)$$

Antisymmetry follows from

$$[[\alpha], [\beta]] + [[\beta], [\alpha]] = [d(\delta\alpha \cdot \delta\beta)] = 0\,.$$

For the Jacobi identity note that, by (73), the jacobiator equals to

$$\sum_{\mathrm{cycl}} [[[\alpha_1], [\alpha_2]], [\alpha_3]] = \Big[\sum_{\mathrm{cycl}} \{\delta\alpha_1, \delta\alpha_2\} \cdot d\delta\alpha_3\Big]\,.$$

This is zero by (74), because by (24) we have

$$\begin{aligned} \delta(\delta\alpha_1 \cdot d\delta\alpha_2 \wedge d\delta\alpha_3) &= \{\delta\alpha_1, \delta\alpha_2\} d\delta\alpha_3 - \{\delta\alpha_1, \delta\alpha_3\} d\delta\alpha_2 - \delta\alpha_1 d\{\delta\alpha_2, \delta\alpha_3\} \\ &= \sum_{\mathrm{cycl}} \{\delta\alpha_1, \delta\alpha_2\} d\delta\alpha_3 - d(\delta\alpha_1 \{\delta\alpha_2, \delta\alpha_3\})\,. \end{aligned}$$

The Lie algebra homomorphism π is surjective by Lemma 2.11. The kernel of π consists of those $[\alpha]$ in $\Omega^1(X)/\delta\Omega^2(X)$ for which $\delta\alpha$ is locally constant, hence $\mathrm{Ker}(\pi) = \mathrm{Ker}(d \circ \delta)/\delta\Omega^2(X)$.

The Lie algebra $\mathrm{ham}(X)$ has trivial center, because $[X_f, X_g] = 0$ for all $X_g \in \mathrm{ham}(X)$ implies that $i_{X_f} dg$ is constant for all $g \in C^\infty(X)$, hence $X_f = 0$. Since π maps the center $\mathfrak{z}$ of $\Omega^1(X)/\delta\Omega^2(X)$ to the (trivial) center of $\mathrm{ham}(X)$, we have $\mathfrak{z} \subseteq \mathrm{Ker}(\pi)$. Since $d \circ \delta\alpha = 0$ implies that $[\alpha] \in \Omega^1(X)/\delta\Omega^2(X)$ is central, the converse inclusion follows. $\square$

Proposition 5.4. *Let X be a symplectic manifold. Then $\Omega^1(X)/\delta\Omega^2(X)$, equipped with the Lie bracket* (69)*, is a perfect Lie algebra.*

Proof. First, suppose that X is connected, and $[\alpha] \in \Omega^1(X)/\delta\Omega^2(X)$ is of the form $[\alpha] = [fdg]$, with $f, g \in C^\infty(X)$. If X is noncompact, then $\delta : \Omega^1(X) \to C^\infty(X)$ is surjective by Lemma 2.11 and we can choose $\phi, \gamma \in \Omega^1(X)$ with $\delta\phi = f$ and $\delta\gamma = g$, so that $[\alpha] = [\delta\phi \cdot d\delta\gamma] = [[\phi], [\gamma]]$ is a commutator. If, on the other hand, X is compact, then $\mathrm{Im}(\delta)$ is equal to $C^\infty_{c,0}(X)$, the space of zero integral functions, by Lemma 2.11. If we write $f = f_0 + c_f$ with $f_0 \in C^\infty_{c,0}(X)$ and $c_f \in \mathbb{R}$, and similarly $g = g_0 + c_g$, then $[fdg] = [(f_0 + c_f)dg_0] = -[df_0 g_0] = [f_0 dg_0]$. Since we can choose $\phi, \gamma \in \Omega^1(X)$ with $f_0 = \delta\phi$ and $g_0 = \delta\gamma$, we see that $[\alpha] = [\delta\phi \cdot d\delta\gamma] = [[\phi], [\gamma]]$ is a commutator.

It suffices to show, then, that $\alpha \in \Omega^1(X)$ can be written as a finite sum of elements of the form fdg. As in the proof of Prop. 3.3, we use dimension theory ([HW41, Thm. V1]) to find a cover $U_{k,r}$ of X by open Darboux coordinate patches $U_{k,r}$, where $k \in \mathbb{N}$ is a countable index, $r \in \{1, \ldots, 2n+1\}$ is a finite index, and $U_{k,r} \cap U_{k',r} = \emptyset$ for all $k \neq k'$. Using a partition of unity, we write $\alpha = \sum_{k=1}^{\infty} \sum_{r=1}^{2n+1} \alpha^{k,r}$ with $\alpha^{k,r} \in \Omega^1_c(U_{k,r})$. The intersection properties of the $U_{k,r}$ ensure that in a single point $x \in X$, the sum has at most $2n+1$ nonzero terms. In local coordinates x^μ, we have $\alpha^{k,r} = \sum_{\mu=1}^{2n} \alpha^{k,r}_\mu dx^\mu$. Choosing compactly supported functions $\xi^{k,r,\mu}$ on $U_{k,r}$ that agree with x^μ on the support of $\alpha^{k,r}_\mu$, we see that $\alpha^{k,r} = \sum_{\mu=1}^{2n} \alpha^{k,r}_\mu d\xi^{k,r,\mu}$ is of the required form. If we set $f^r_\mu := \sum_{k \in \mathbb{N}} \alpha^{k,r}_\mu$ and $g^{r,\mu} := \sum_{k\in\mathbb{N}} \xi^{k,r,\mu}$, then because $\alpha^{k,r}_\mu d\xi^{k',r,\nu} = 0$ for $k \neq k'$, we have

$$\sum_{r=1}^{2n+1} \sum_{\mu=1}^{2n} f^r_\mu dg^{r,\mu} = \sum_{r=1}^{2n+1} \sum_{\mu=1}^{2n} \sum_{k\in\mathbb{N}} \alpha^{k,r}_\mu d\xi^{k,r,\mu} = \sum_{r=1}^{2n+1} \sum_{k\in\mathbb{N}} \alpha^{k,r} = \alpha\,.$$

It follows that for X connected, $[\alpha]$ is a sum of at most $2n(2n+1)$ commutators $[\phi^r_\mu]$ and $[\gamma^{r,\mu}]$, with $\delta\phi^r_\mu = f^r_\mu$ and $\delta\gamma^{r,\mu} = g^{r,\mu}$ if X is noncompact, and $\delta\phi^r_\mu = f^r_{\mu 0}$ and $\delta\gamma^{r,\mu} = g^{r,\mu}_0$ if X is compact.

If X is not connected, then we find $\phi^{r,x}_\mu$ and $\gamma^{r,\mu,x}$ as above for every connected component X_x of X, so that on X_x, we have

$$[\alpha|_{X_x}] = \sum_{r=1}^{2n+1} \sum_{\mu=1}^{2n} [\delta\phi^{r,x}_\mu d\delta\gamma^{r,\mu,x}]\,.$$

If we set $\phi^r_\mu = \sum_x \phi^{r,x}_\mu$ and $\gamma^{r,\mu} := \sum_x \gamma^{r,\mu,x}$, then since $\delta\phi^{r,x}_\mu d\delta\gamma^{r,\mu,x'} = 0$ for $X_x \neq X_{x'}$, we have

$$[\alpha] = \sum_{r=1}^{2n+1} \sum_{\mu=1}^{2n} [[\phi^r_\mu], [\gamma^{r,\mu}]]$$

as a sum of at most $2n(2n+1)$ commutators. $\square$

Remark 5.5. It follows from the previous proof that if X is of dimension $2n$, then every element of $\Omega^1(X)/\delta\Omega^2(X)$ can be written as a sum of at most $2n(2n+1)$ commutators, cf. Rk. 3.4.

Recall from Sec. 2.3 that a linearly split central extension is called *universal* for $\mathfrak{a}$ if it maps to every linearly split central $\mathfrak{a}$-extension, and *universal* if it is $\mathfrak{a}$-universal for every locally convex space $\mathfrak{a}$.

Theorem 5.6. *Let X be a connected symplectic manifold. Then the continuous central extension*

$$\mathfrak{z} \xrightarrow{\iota} \Omega^1(X)/\delta\Omega^2(X) \xrightarrow{\pi} \mathrm{ham}(X) \tag{75}$$

of Fréchet Lie algebras (cf. Prop. 5.1) is linearly split, and $\mathfrak{a}$-universal for finite dimensional vector spaces $\mathfrak{a}$. If, moreover, $H^{2n-1}_{\mathrm{dR}}(X)$ is finitely generated, then it is the universal central extension of $\mathrm{ham}(X)$.

Proof. By Prop. 2.9 and Thm. 4.17, every central $\mathbb{R}$-extension of $\mathrm{ham}(X)$ is isomorphic to $\mathbb{R} \oplus_{\psi_\alpha} \mathrm{ham}(X)$, with Lie bracket

$$\big[(a, X_f), (b, X_g)\big] = \Big(\psi_\alpha(X_f, X_g), X_{\{f,g\}}\Big)\,. \tag{76}$$

The class $[\psi_\alpha] \in H^2(\mathrm{ham}(X), \mathbb{R})$ is determined by $\alpha \in Z^1_c(X)/d\Omega^0_{c,0}(X)$, cf. eqn. (62). Using

$$f \cdot (i_{X_g}\alpha)\omega^n/n! \stackrel{(26)}{=} -\alpha \wedge (fdg) \wedge \omega^{n-1}/(n-1)! \stackrel{(20)}{=} -\alpha \wedge *(fdg)\,,$$

we rewrite the cocycle ψ_α as

$$\psi_\alpha(X_f, X_g) = \int_X f \cdot (i_{X_g}\alpha)\omega^n/n! = S_\alpha(fdg)\,,$$

with $S_\alpha \colon \Omega^1(X) \to \mathbb{R}$ given by

$$S_\alpha(\beta) := -\int_X \alpha \wedge *\beta\,.$$

Note that S_α factors through a functional $\overline{S}_\alpha \colon \Omega^1(X)/\delta\Omega^2(X) \to \mathbb{R}$, because for $F \in \Omega^2(X)$ we have $\int_X \alpha \wedge *\delta F = \int_X \alpha \wedge d * F = \int_X d(\alpha \wedge *F) = 0$.

For every central $\mathbb{R}$-extension of $\mathrm{ham}(X)$, we thus obtain a continuous linear map

$$\phi \colon \Omega^1(X)/\delta\Omega^2(X) \to \mathbb{R} \oplus_{\psi_\alpha} \mathrm{ham}(X)$$

defined by

$$\phi([\beta]) := (\overline{S}_\alpha([\beta]), \pi(\beta))\,. \tag{77}$$

This is a Lie algebra homomorphism because $[[\beta], [\beta']] = [\delta\beta \cdot d\delta\beta']$ by definition of the Lie bracket on $\Omega^1(X)/\delta\Omega^2(X)$, so

$$\overline{S}_\alpha([[\beta], [\beta']]) = S_\alpha(\delta\beta \cdot d\delta\beta') = \psi_\alpha(X_{\delta\beta}, X_{\delta\beta'})\,.$$

The homomorphism ϕ is unique with the property that the $\mathrm{ham}(X)$-valued component coincides with π. Indeed, if $\phi' \colon \Omega^1(X)/\delta\Omega^2(X) \to \mathbb{R} \oplus_{\psi_\alpha} \mathrm{ham}(X)$

is another such homomorphism, then $\phi - \phi'$ is a Lie algebra homomorphism $\Omega^1(X)/\delta\Omega^2(X) \to \mathbb{R} \oplus \{0\}$, hence trivial because $\Omega^1(X)/\delta\Omega^2(X)$ is perfect (cf. Prop. 5.4). This shows that the continuous central extension (75) is $\mathbb{R}$-universal, hence universal for finite dimensional vector spaces $\mathfrak{a}$ by [Nee02b, Lemma 1.12].

A linear splitting of (75) exists because the de Rham complex $(\Omega^\bullet(X), d)$, and hence the isomorphic canonical complex $(\Omega^\bullet(X), \delta)$, is split and cosplit as a complex of Fréchet spaces [Pal72, Prop. 5.4]. A linear splitting $\mathrm{ham}(X) \to C^\infty(X)$ corresponds to a splitting of the canonical complex in degree 0, and a splitting $C^\infty(X) \to \Omega^1(X)$ to a splitting in degree 1. If X is compact, then a splitting of the de Rham complex can also be obtained using Hodge theory.

If $H^{2n-1}_{\mathrm{dR}}(X)$ is finite dimensional, then so is $H^2(\mathrm{ham}(X), \mathbb{R})$, so a universal central extension with finite dimensional kernel exists by [Nee02b, Lem. 2,7, Cor. 2.13]. Since this universal central extension and the extension (75) are both linearly split as well as $\mathfrak{a}$-universal for finite dimensional $\mathfrak{a}$, they are isomorphic, and (75) is universal. $\square$

Remark 5.7. If we identify $\Omega^1(X)/\delta\Omega^2(X)$ with $\Omega^{2n-1}(X)/d\Omega^{2n-2}(X)$ according to Remark 5.3, the morphism (77) to the central extension $\mathbb{R} \oplus_{\psi_\alpha} \mathrm{ham}(X)$ with cocycle ψ_α defined in (62) takes the particularly simple form

$$[\gamma] \mapsto (-\int_X \alpha \wedge \gamma, X_f)\,,$$

where the function f is given by $f\omega^n/n! = d\gamma$.

From the above theorem, one readily derives the universal central extension of the Poisson algebra $C^\infty(X)$ for a noncompact connected manifold X. If X is compact, no universal central extension exists because $C^\infty(X)$ is not perfect (cf. Prop. 3.1).

Corollary 5.8. *Let X be a noncompact connected symplectic manifold. Then the continuous central extension*

$$H_1^{\mathrm{can}}(X) \to \Omega^1(X)/\delta\Omega^2(X) \xrightarrow{\delta} C^\infty(X)$$

is linearly split and $\mathfrak{a}$-universal for finite dimensional spaces $\mathfrak{a}$. If, moreover, $H_1^{\mathrm{can}}(X) \simeq H^{2n-1}_{\mathrm{dR}}(X)$ is finitely generated, then it is the universal central extension of $C^\infty(X)$.

Proof. By eqn. (39), every $\mathfrak{a}$-valued 2-cocycle ψ on $C^\infty(X)$ vanishes on the constant functions. It follows that every linearly split continuous central extension $\mathfrak{a} \to \mathfrak{g}^\sharp \xrightarrow{\mathrm{pr}} C^\infty(X)$ of $C^\infty(X)$ defines, by concatenation with the map $C^\infty(X) \to \mathrm{ham}(X)$ that takes a hamilton function to the corresponding hamiltonian vector field, a linearly split continuous central extension of $\mathrm{ham}(X)$. If either $\mathfrak{a}$ or $H^{2n-1}_{\mathrm{dR}}(X)$ is finite dimensional, then Thm. 5.6 yields a unique continuous homomorphism $\phi\colon \Omega^1(X)/\delta\Omega^2(X) \to \mathfrak{g}^\sharp$ such that the big triangle in the following diagram commutes:

$$\begin{tikzcd}
& & \Omega^1(X)/\delta\Omega^2(X) \arrow[dl, "\phi"'] \arrow[d, "\delta"] \arrow[dr, "\pi"] & \\
\mathfrak{a} \arrow[r] & \mathfrak{g}^\sharp \arrow[r, "\mathrm{pr}"] & C^\infty(X) \arrow[r] & \mathrm{ham}(X)\,.
\end{tikzcd}$$

Since δ is the unique map which makes the right hand triangle commute, we have $\mathrm{pr}\circ\phi = \delta$. For noncompact X, the cokernel $H_0^{\mathrm{can}}(X) \simeq H^{2n}_{\mathrm{dR}}(X)$ of the Lie algebra homomorphism $\delta\colon \Omega^1(X)/\delta\Omega^2(X) \to C^\infty(X)$ vanishes, so δ is surjective with kernel $H_1^{\mathrm{can}}(X)$. $\square$

Remark 5.9. The Lie algebra of hamiltonian vector fields $\mathrm{ham}(X)$ is perfect for any symplectic manifold X. Indeed, it is the image by π of the perfect Lie algebra $\Omega^1(X)/\delta\Omega^2(X)$ by the Lie algebra homomorphism π. The Poisson Lie algebra $C^\infty(X)$ in the noncompact case and $C^\infty_0(X)$ in the compact case are also perfect, as images of the perfect Lie algebra $\Omega^1(X)/\delta\Omega^2(X)$ by the Lie algebra homomorphism δ. Thus we recover Corollary 3.7, as well as the 'perfect part' of Corollary 3.2 and Proposition 3.3.

Singular cocycles and Roger cocycles revisited. The universal central extension $\Omega^1(X)/\delta\Omega^2(X)$ of $\mathrm{ham}(X)$ yields a straightforward way to decide when the Roger cocycles ψ_α and singular cocycles ψ_N defined on page 5 are cohomologous.

By the universal property of (70) for a connected, symplectic manifold X, continuous 2-cocycles ψ of $\mathrm{ham}(X)$ correspond bijectively to continuous linear functionals S on $\Omega^1(X)/\delta\Omega^2(X)$ by

$$\psi_S(X_f, X_g) = S([f dg])\,,$$

and two cocycles are cohomologous, i.e. $\psi_S \sim \psi_{S'}$, if and only if $S - S'$ vanishes on the center $\mathfrak{z} = \mathrm{Ker}(d\circ\delta)/\delta\Omega^2(X)$.

The singular 2-cocycles ψ_N come from $(2n-1)$-cycles $N = \sum_{i=1}^k c_i\sigma_i$ in the singular homology of X, with real coefficients c_i and piecewise smooth maps $\sigma_i\colon \Delta^{2n-1} \to X$. The functional $S_N([\beta]) = \int_N *\beta$ on $\Omega^1(X)/\delta\Omega^2(X)$ yields the singular 2-cocycle

$$\psi_N(X_f, X_g) = S_N([f dg]) = \int_N f dg \wedge \omega^{n-1}/(n-1)!\,. \tag{78}$$

If X is compact, then $\psi_N \sim \psi_{N'}$ if and only if $N \sim N'$, in the sense that $N - N' = \partial M$ for a singular $2n$-chain M in X with real coefficients. Indeed, $\psi_N \sim \psi_{N'}$ if and only if $\int_{N-N'} *\beta = 0$ for all $\beta \in \Omega^1(X)$ with $d\delta\beta = 0$. For X compact, $d\delta\beta = 0$ is equivalent to $d(*\beta) = 0$, so by Poincaré duality, $\psi_N \sim \psi_{N'}$ if and only if $N - N' = \partial M$. In this case we get an alternative description of $H^2(\mathrm{ham}(X), \mathbb{R})$ in terms of singular homology. The correspondence $[N] \mapsto [\psi_N]$ yields and isomorphism between $H_{2n-1}(X, \mathbb{R})$ and $H^2(\mathrm{ham}(X), \mathbb{R})$.

If X is noncompact, then $\psi_N \sim \psi_{N'}$ if and only if $N - N' = \partial M$, with the additional requirement that M has zero symplectic volume, $\int_M \omega^n/n! = 0$. Indeed, for X noncompact, $d\delta\beta = 0$ is equivalent to $d(*\beta) = c\omega^n/n!$ for some $c \in \mathbb{R}$. From $c = 0$, we find $N - N' = \partial M$, and $c \neq 0$ yields the further requirement $\int_M \omega^n/n! = 0$.

The Roger 2-cocycles ψ_α come from compactly supported, closed 1-forms $\alpha \in \Omega^1_c(X)$,

$$\psi_\alpha(X_f, X_g) = \int_X f(i_{X_g}\alpha)\omega^n/n!\,, \tag{79}$$

via the functionals $S_\alpha([\beta]) := \int_X *\beta \wedge \alpha$. Two such cocycles are cohomologous, i.e. $\psi_\alpha \sim \psi_{\alpha'}$, if and only if $\alpha - \alpha' = dh$ with $\int_X h\omega^n/n! = 0$. This yields the isomorphism $H^2(\mathrm{ham}(X), \mathbb{R}) \simeq Z^1_c(X)/d\Omega^0_{c,0}(X)$ of Theorem 4.17.

In the same vein, we have $\psi_\alpha \sim \psi_N$ if and only if $\int_N \gamma = \int_X \alpha \wedge \gamma$ for every $\gamma \in \Omega^{2n-1}(X)$ with $\delta d\gamma = 0$. In the compact case, this reduces to the requirement is that $[N] \in H_{2n-1}(X, \mathbb{R})$ be Poincaré dual to $[\alpha] \in H^1_{\mathrm{dR},c}(X)$.

6 Central extensions of $\mathrm{sp}(X)$

Let X be a connected symplectic manifold. In order to determine $H^2(\mathrm{sp}(X), \mathbb{R})$, we observe that $\mathrm{ham}(X)$ is a perfect, closed, complemented ideal in $\mathrm{sp}(X)$ with abelian quotient $\mathrm{sp}(X)/\mathrm{ham}(X) \simeq H^1_{\mathrm{dR}}(X)$.

6.1 A 5-term exact sequence

If $\mathfrak{h}$ is a perfect, closed, complemented ideal in a Fréchet Lie algebra $\mathfrak{g}$, then it is shown in [Viz06, Thm. 2.3] that the exact sequence

$$0 \to \mathfrak{h} \xrightarrow{\iota} \mathfrak{g} \xrightarrow{p} \mathfrak{g}/\mathfrak{h} \to 0 \tag{80}$$

of Fréchet Lie algebras gives rise to a 5-term exact sequence

$$0 \to H^2(\mathfrak{g}/\mathfrak{h}, \mathbb{R}) \xrightarrow{p^*} H^2(\mathfrak{g}, \mathbb{R}) \xrightarrow{\iota^*} H^2(\mathfrak{h}, \mathbb{R})^{\mathfrak{g}} \xrightarrow{T} H^3(\mathfrak{g}/\mathfrak{h}, \mathbb{R}) \xrightarrow{p^*} H^3(\mathfrak{g}, \mathbb{R}) \tag{81}$$

in continuous Lie algebra cohomology.

The transgression map $T\colon H^2(\mathfrak{h}, \mathbb{R})^{\mathfrak{g}} \to H^3(\mathfrak{g}/\mathfrak{h}, \mathbb{R})$ is defined as follows. If $[\psi] \in H^2(\mathfrak{h}, \mathbb{R})$ is $\mathfrak{g}$-invariant, then for each $v \in \mathfrak{g}$, there exists a unique $\theta_v \in \mathfrak{h}'$ such that $L_v \psi = \mathtt{d}\theta_v$. Moreover, the map $\mathfrak{g} \to \mathfrak{h}'$ given by $v \mapsto \theta_v$ is linear, and the corresponding map $\theta\colon \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$ continuous [Viz06, Lemma 2.1]. If $\psi'\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ is a skew-symmetric continuous extension of θ, then $\mathtt{d}\psi'$ induces a 3-cocycle $\overline{\mathtt{d}\psi'}$ on $\mathfrak{g}/\mathfrak{h}$, whose class $[\overline{\mathtt{d}\psi'}] \in H^3(\mathfrak{g}/\mathfrak{h})$ does not depend on the choice of ψ'. The transgression is defined as $T([\psi]) := [\overline{\mathtt{d}\psi'}]$.

Remark 6.1. Since $H^1(\mathfrak{h}, \mathbb{R}) = 0$, the transgression map T coincides with the map d_3 for the Hochschild-Serre spectral sequence associated to the filtration

$$F^p C^{p+q}(\mathfrak{g}) = \{\sigma \in C^{p+q}(\mathfrak{g}) : i_{H_1} \dots i_{H_{q+1}} \sigma = 0, \text{ for all } H_i \in \mathfrak{h}\}\,.$$

From (81) with $\mathfrak{h} = \mathrm{ham}(X)$ and $\mathfrak{g} = \mathrm{sp}(X)$, we find

$$H^2(\mathrm{sp}(X), \mathbb{R}) \simeq \textstyle\bigwedge^2 H^1_{\mathrm{dR}}(X)^* \oplus \mathrm{Ker}(T)\,, \tag{82}$$

where the map $\bigwedge^2 H^1_{\mathrm{dR}}(X)^* \to H^2(\mathrm{sp}(X), \mathbb{R})$ takes $\gamma \in \bigwedge^2 H^1_{\mathrm{dR}}(X)^*$ to the class of $\psi_\gamma(v, w) := \gamma([i_v\omega], [i_w\omega])$. For example, if X is compact, the alternating pairing $\gamma([\alpha], [\beta]) = \int_X \alpha \wedge \beta \wedge \omega^{n-1}/(n-1)!$ gives rise to the Lie algebra cocycle described in [ILM06].

6.2 The $\mathrm{sp}(X)$-invariants in $H^2(\mathrm{ham}(X), \mathbb{R})$

In order to determine the remaining part $\mathrm{Ker}(T)$ of the second cohomology group of $\mathrm{sp}(X)$, we must first determine the $\mathrm{sp}(X)$-invariant part of $H^2(\mathrm{ham}(X), \mathbb{R})$, and then calculate the transgression map T. Recall from Theorem 4.17 that every class $[\psi] \in H^2(\mathrm{ham}(X), \mathbb{R})$ has a representative of the form

$$\psi_\alpha(X_f, X_g) = \int_X f\alpha(X_g)\omega^n/n!$$

for a closed compactly supported 1-form $\alpha \in \Omega^1_c(X)$, and that $[\psi_\alpha] = [\psi_{\alpha'}]$ if and only if $\alpha - \alpha' = dh$ with $\langle h \rangle = 0$, where

$$\langle h \rangle := \int_X h\omega^n/n!\,. \tag{83}$$

Proposition 6.2. *Let X be a connected symplectic manifold. Then the action of the symplectic vector fields* $\mathrm{sp}(X)$ *on* $H^2(\mathrm{ham}(X),\mathbb{R})$ *is given by*

$$L_v[\psi_\alpha] = -\langle i_v\alpha\rangle[\psi_{KS}]\,,$$

where v is an element of $\mathrm{sp}(X)$ *and* $[\psi_{KS}]$ *is the Kostant-Souriau class* (34).

Proof. First we rewrite the cocycle ψ_α, using (26), as

$$\psi_\alpha(X_f, X_g) = \int_X f dg \wedge \alpha \wedge \omega^{n-1}/(n-1)!\,.$$

It follows that $(L_v\psi_\alpha)(X_f, X_g) := -\psi_\alpha(L_vX_f, X_g) - \psi_\alpha(X_f, L_vX_g)$ satisfies for all $v \in \mathrm{sp}(X)$

$$\begin{aligned}(L_v\psi_\alpha)(X_f, X_g) &= -\int_X L_v(fdg) \wedge \alpha \wedge \omega^{n-1}/(n-1)! \\ &= \int_X f dg \wedge d\big(\alpha(v)\omega^{n-1}\big)/(n-1)! \\ &= -\int_X \alpha(v) df \wedge dg \wedge \omega^{n-1}/(n-1)!\,,\end{aligned}$$

so that by equation (30), we have

$$(L_v\psi_\alpha)(X_f, X_g) = -\int_X \alpha(v)\{f, g\}\omega^n/n!\,. \tag{84}$$

For $x \in X$ and $\alpha \in \Omega^1_c(X)$ closed, we define the continuous map $\theta_\alpha \colon \mathrm{sp}(X) \to \mathrm{ham}(X)'$ by

$$\theta_\alpha(v)(X_f) := \int_X \alpha(v)(f - f_x)\omega^n/n!, \tag{85}$$

where $f - f_x$ is the unique Hamiltonian function of X_f that vanishes at x. Since the coboundary of $\theta_\alpha(v)$ is a linear combination of $L_v\psi_\alpha$ and ψ_{KS}, namely

$$\begin{aligned}\mathtt{d}(\theta_\alpha(v))(X_f, X_g) &= -\theta_\alpha(v)(X_{\{f,g\}}) \qquad (86)\\ &= -\int_X \alpha(v)(\{f, g\} - \{f, g\}_x)\omega^n/n! \\ &= \langle\alpha(v)\rangle\psi_{KS}(X_f, X_g) + L_v\psi_\alpha(X_f, X_g)\,,\end{aligned}$$

we have $L_v[\psi_\alpha] = -\langle\alpha(v)\rangle[\psi_{KS}]$. ∎

This reveals a dichotomy between the compact and noncompact case: if X is compact, then $[\psi_{KS}]$ is zero by Corr. 3.6, so every class in $\mathrm{ham}(X)$ is automatically $\mathrm{sp}(X)$-invariant.

Corollary 6.3. *Let X be a compact connected symplectic manifold. Then* $\mathrm{sp}(X)$ *acts trivially on the continuous second cohomology* $H^2(\mathrm{ham}(X),\mathbb{R})$.

If, on the other hand, X is noncompact, then the $\mathrm{sp}(X)$-invariant part of $H^2(\mathrm{ham}(X), \mathbb{R})$ can be quite small. Indeed, in terms of the alternating pairing

$$(\,\cdot\,,\,\cdot\,)\colon H^1_{\mathrm{dR,c}}(X) \times H^1_{\mathrm{dR}}(X) \to \mathbb{R}$$

defined by

$$([\alpha], [\beta]) := \int_X \alpha \wedge \beta \wedge \omega^{n-1}/(n-1)!\,, \tag{87}$$

we have the following result.

Corollary 6.4. *Let X be a noncompact connected symplectic manifold. Then the class $[\psi_\alpha] \in H^2(\mathrm{ham}(X), \mathbb{R})$ is annihilated by $\mathrm{sp}(X)$ if and only if $[\alpha] \in H^1_{\mathrm{dR,c}}(X)$ is in the kernel of the alternating pairing* (87). *In particular, we have $\mathbb{R}[\psi_{KS}] \subseteq H^2(\mathrm{ham}(X), \mathbb{R})^{\mathrm{sp}(X)}$.*

Proof. Since $[\psi_{KS}]$ is nonzero by Corollary 3.6, Proposition 6.2 shows that $[\psi_\alpha]$ is annihilated by $\mathrm{sp}(X)$ if and only if $\langle \alpha(v) \rangle$ vanishes for all $v \in \mathrm{sp}(X)$. Because $\langle \alpha(v) \rangle = ([\alpha], [i_v\omega])$ by (26) and $v \mapsto [i_v\omega]$ is surjective onto $H^1_{\mathrm{dR}}(X)$, we have that $\langle \alpha(v) \rangle = 0$ for all $v \in \mathrm{sp}(X)$ if and only if $[\alpha]$ is in the kernel of the alternating pairing (87).

Since $[\psi_{KS}]$ is equal to $[\psi_{dh}]$ for $h \in \Omega^0_c(X)$ with $\langle h \rangle = 1$, $[\alpha] = [dh] = 0$ is in the kernel, so $[\psi_{KS}]$ is always $\mathrm{sp}(X)$-invariant. □

Remark 6.5. By definition, a symplectic manifold is Lefschetz if the map

$$H^1_{\mathrm{dR}}(X) \to H^{2n-1}_{\mathrm{dR}}(X)\colon [\alpha] \mapsto [\omega^{n-1} \wedge \alpha]$$

is an isomorphism. Since the intersection pairing is then nondegenerate, it follows that for a noncompact Lefschetz manifold X, the only invariant class is $[\psi_{KS}]$,

$$H^2(\mathrm{ham}(X), \mathbb{R})^{\mathrm{sp}(X)} = \mathbb{R}[\psi_{KS}]\,.$$

In view of the different character of $H^2(\mathrm{ham}(X), \mathbb{R})$ for compact and noncompact manifolds, we will treat these two cases separately.

6.3 Transgression for compact manifolds

We first determine the continuous second Lie algebra cohomology of $\mathrm{sp}(X)$ in the case that X is a *compact* connected manifold, and return to noncompact manifolds in the next section. In view of eqn. (82), Cor. 6.3, and Thm. 4.17, this amounts to determining the transgression map $T\colon H^1_{\mathrm{dR}}(X) \to \bigwedge^3 H^1_{\mathrm{dR}}(X)^*$. This was done in [Viz06], using the characteristic class one obtains from the Weil homomorphism applied to an invariant bilinear form on $\mathrm{ham}(X)$.

We now give a direct construction of the transgression map, which avoids the use of characteristic classes, and therefore generalizes more easily to noncompact manifolds, where an invariant bilinear form is not available.

Proposition 6.6. *For X a compact connected symplectic manifold, the transgression map in the five term exact sequence* (81) *is given by*

$$T\colon H^1_{\mathrm{dR}}(X) \to \textstyle\bigwedge^3 H^1_{\mathrm{dR}}(X)^*$$

$$T(a)(b_1, b_2, b_3) = (a, b_1, b_2, b_3) - \frac{1}{\mathrm{vol}(X)}\Big(\sum_{\mathrm{cycl}} (a, b_1)(b_2, b_3)\Big) \tag{88}$$

with the alternating 4-form on $H^1_{\mathrm{dR}}(X)$ defined by

$$([\alpha],[\beta_1],[\beta_2],[\beta_3]) = \int_X \alpha \wedge \beta_1 \wedge \beta_2 \wedge \beta_3 \wedge \omega^{n-2}/(n-2)! \tag{89}$$

and the skew symmetric 2-form $(\,\cdot\,,\,\cdot\,)$ on $H^1_{\mathrm{dR}}(X)$ as in eqn. (87).

Note that one can always rescale ω so that $\mathrm{vol}(X) = 1$, which was implicitly done in the derivation of [Viz06, eqn. (8)].

Proof. By Corollary 6.3, we have

$$H^2(\mathrm{ham}(X),\mathbb{R})^{\mathrm{sp}(X)} = H^2(\mathrm{ham}(X),\mathbb{R}),$$

and by Theorem 4.17, $H^2(\mathrm{ham}(X),\mathbb{R})$ can be identified with $H^1_{\mathrm{dR}}(X)$ by mapping $[\alpha] \in H^1_{\mathrm{dR}}(X)$ to the class $[\psi_\alpha] \in H^2(\mathrm{ham}(X),\mathbb{R})$ of the cocycle ψ_α described in eqn. (62). The quotient $\mathrm{sp}(X)/\mathrm{ham}(X) = H^1_{\mathrm{dR}}(X)$ is an abelian Lie algebra, so the transgression is a map $T\colon H^1_{\mathrm{dR}}(X) \to \bigwedge^3 H^1_{\mathrm{dR}}(X)^*$.

Since X is compact, we can identify $\mathrm{ham}(X)$ with the Lie algebra $C^\infty_{c,0}(X)$ of zero integral functions by mapping X_f to its unique zero integral Hamilton function $f - \frac{1}{\mathrm{vol}(X)}\langle f\rangle$. Because the Hamiltonian function $\{f,g\}$ of $[X_f, X_g]$ has zero integral, we see from eqn. (84) that $L_v\psi_\alpha = \mathtt{d}(\Theta_\alpha(v))$, with the 1-cochain $\Theta_\alpha : \mathrm{sp}(X) \to \mathrm{ham}(X)'$ given by

$$\Theta_\alpha(v)(X_f) = \int_X \alpha(v)\left(f - \tfrac{1}{\mathrm{vol}(X)}\langle f\rangle\right)\omega^n/n!,$$

where $\langle f\rangle$ is given by (83) and $\mathrm{vol}(X) = \langle 1\rangle$ is the symplectic volume of X. To compute the transgression map, we proceed as outlined in Section 6.1. We choose a 2-cochain ψ'_α on $\mathrm{sp}(X)$ that extends $\Theta_\alpha\colon \mathrm{sp}(X) \times \mathrm{ham}(X) \to \mathbb{R}$, and compute its differential. Using Lemma 2.12 and the fact that the Lie bracket of two symplectic vector fields v_1 and v_2 is Hamiltonian with Hamilton function $\omega(v_1,v_2)$, we compute the differential of ψ'_α as

$$\begin{aligned}
(\mathtt{d}\psi'_\alpha)(v_1,v_2,v_3) &= \sum_{cycl}\psi'_\alpha(v_1,[v_2,v_3]) = \sum_{\mathrm{cycl}}\Theta_\alpha(v_1)([v_2,v_3]) \\
&= \sum_{\mathrm{cycl}}\int_X \alpha(v_1)\left(\omega(v_2,v_3) - \tfrac{1}{\mathrm{vol}(X)}\langle\omega(v_2,v_3)\rangle\right)\omega^n/n! \\
&= \int_X \alpha \wedge i_{v_1}\omega \wedge i_{v_2}\omega \wedge i_{v_3}\omega \wedge \omega^{n-2}/(n-2)! \\
&\quad -\frac{1}{\mathrm{vol}(X)}\sum_{\mathrm{cycl}}\langle\alpha(v_1)\rangle\langle\omega(v_2,v_3)\rangle \\
&= ([\alpha],[i_{v_1}\omega],[i_{v_2}\omega],[i_{v_3}\omega]) \\
&\quad -\frac{1}{\mathrm{vol}(X)}\sum_{\mathrm{cycl}}([\alpha],[i_{v_1}\omega])([i_{v_2}\omega],[i_{v_3}\omega])\,.
\end{aligned}$$

With the expression $T([\psi_\alpha]) := [\overline{\mathtt{d}\psi'_\alpha}]$ of the transgression map, the conclusion follows. $\square$

Knowing this explicit expression for the transgression map, equation (82) yields the following complete description for $H^2(\mathrm{sp}(X), \mathbb{R})$.

Theorem 6.7. *Let X be a compact connected symplectic manifold. Then we have an isomorphism*

$$H^2(\mathrm{sp}(X), \mathbb{R}) \simeq \bigwedge^2 (H^1_{\mathrm{dR}}(X)^*) \oplus \mathrm{Ker}(T),$$

where $\mathrm{Ker}(T) \subseteq H^1_{\mathrm{dR}}(X)$ *is the set of classes $a \in H^1_{\mathrm{dR}}(X)$ such that*

$$(a, b_1, b_2, b_3) = \frac{1}{\mathrm{vol}(X)} \sum_{\mathrm{cycl}} (a, b_1)(b_2, b_3)$$

for all $b_1, b_2, b_3 \in H^1_{\mathrm{dR}}(X)$.

We give a few examples to illustrate that, in concrete situations, the kernel of the transgression can often be determined explicitly.

Example 6.8 (Tori)**.** For the symplectic $2n$-torus $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$, the transgression map vanishes, because $\mathrm{sp}(\mathbb{T}^{2n}) \simeq \mathrm{ham}(\mathbb{T}^{2n}) \rtimes \mathbb{R}^{2n}$ is a semidirect product, cf. [Viz06]. Therefore, all Roger cocycles extend to $\mathrm{sp}(\mathbb{T}^{2n})$, and $H^2(\mathrm{sp}(\mathbb{T}^{2n}), \mathbb{R}) \simeq \bigwedge^2 (\mathbb{R}^{2n}) \oplus \mathbb{R}^{2n}$.

Example 6.9 (Compact surfaces)**.** For a surface Σ_g of genus $g \geq 2$, the transgression map is injective; although the 4-form (89) is zero, the cyclic sum involving the nondegenerate 2-form (87) does not vanish, cf. [Viz06]. Therefore no Roger cocycle can be extended to $\mathrm{sp}(\Sigma_g)$, and $H^2(\mathrm{sp}(\Sigma_g), \mathbb{R}) \simeq \bigwedge^2 \mathbb{R}^{2g}$. For genus 0, we have $H^2(\mathrm{sp}(S^2), \mathbb{R}) = \{0\}$, and for genus 1, we have $H^2(\mathrm{sp}(\mathbb{T}^2), \mathbb{R}) \simeq \bigwedge^2 (\mathbb{R}^2) \oplus \mathbb{R}^2$ by the previous remark.

Remark 6.10 (Compact manifolds with $b_1 < 4$)**.** Since the transgression (88) induces an *alternating* map $\bigwedge^4 H^1_{\mathrm{dR}}(X) \to \mathbb{R}$, it vanishes for dimensional reasons if the first Betti number satisfies $b_1 < 4$. In this case, all Roger cocycles extend to $\mathrm{sp}(X)$, and $H^2(\mathrm{sp}(X), \mathbb{R}) \simeq \bigwedge^2 (\mathbb{R}^{b_1}) \oplus \mathbb{R}^{b_1}$.

Example 6.11. *Thurston's manifold* X^4 was historically the first example of a compact, symplectic manifold which is not Kähler [Thu76]. It is a *nilmanifold*, i.e., a quotient $X = \Gamma \backslash N$ of a 1-connected, nilpotent Lie group N by a discrete, cocompact subgroup Γ. For Thurston's manifold, the nilpotent Lie group is $N = H \times \mathbb{R}$, where H is the *Heisenberg group.* Its Lie algebra is $\mathfrak{n} = \mathfrak{h} \times \mathbb{R}$, where $\mathfrak{h} = \mathrm{Span}\,\langle x, p, h \rangle$ is the *Heisenberg Lie algebra* with $[x, p] = h$ and $[h, x] = [h, p] = 0$, and $\mathbb{R}$ is central in $\mathfrak{n}$ with generator z. We denote by x^*, p^*, h^*, z^* the dual basis of x, p, h, z. Under the inclusion $\iota \colon \bigwedge \mathfrak{n}^* \hookrightarrow \Omega(X^4)$ as left invariant forms, the symplectic form is the image of

$$\omega = h^* \wedge x^* + z^* \wedge p^*\,,$$

normalised so that the symplectic volume $\mathrm{vol}(X^4)$ is 1. By a theorem of Nomizu [Nom54], the inclusion $\iota \colon \bigwedge \mathfrak{n}^* \hookrightarrow \Omega(X^4)$ yields an isomorphism between the Lie algebra cohomology $H^\bullet(\mathfrak{n}, \mathbb{R})$ and the de Rham cohomology $H^\bullet_{\mathrm{dR}}(X^4)$. The Lie algebra differential $\mathtt{d} \colon \bigwedge \mathfrak{n}^* \to \bigwedge \mathfrak{n}^*$ is the derivation with $\mathtt{d}h^* = -x^* \wedge p^*$ and $\mathtt{d}x^* = \mathtt{d}p^* = \mathtt{d}z^* = 0$. From this, one finds that the cohomology $H^\bullet(\mathfrak{n}, \mathbb{R})$

is generated (as a ring) by $[x^*]$, $[p^*]$ and $[z^*]$ in degree 1 and $[h^* \wedge p^*]$, $[h^* \wedge x^*]$ in degree 2, with the single relation $[x^* \wedge p^*] = 0$. The fact that

$$H^1_{\mathrm{dR}}(X^4) = \mathrm{Span}\,\langle [x^*], [p^*], [z^*]\rangle \tag{90}$$

is 3-dimensional led Thurston to his conclusion that X^4 does not admit a Kähler structure. By Remark 6.10, it also implies that every Roger cocycle extends to $\mathrm{sp}(X^4)$, and $H^2(\mathrm{sp}(X^4), \mathbb{R}) \simeq \bigwedge^2(\mathbb{R}^3) \oplus \mathbb{R}^3$.

6.4 Transgression for noncompact manifolds

We now determine the continuous second Lie algebra cohomology of $\mathrm{sp}(X)$ in the case that X is a *noncompact* connected manifold. As in the previous section, this amounts to determining the transgression map

$$T\colon H^2(\mathrm{ham}(X), \mathbb{R})^{\mathrm{sp}(X)} \to \textstyle\bigwedge^3 H^1_{\mathrm{dR}}(X)^*,$$

but in contrast to the compact case, not every class $[\psi_\alpha] \in H^2(\mathrm{ham}(X), \mathbb{R})$ is $\mathrm{sp}(X)$-invariant. By Cor. 6.4, $[\psi_\alpha]$ is $\mathrm{sp}(X)$-invariant if and only if $[\alpha] \in H^1_{\mathrm{dR,c}}(X)$ lies in the kernel of the alternating pairing (87), and an $\mathrm{sp}(X)$-invariant class $[\psi_\alpha]$ extends from $\mathrm{ham}(X)$ to $\mathrm{sp}(X)$ if and only if it lies in the kernel of the transgression map.

Proposition 6.12. *The transgression map*

$$T\colon H^2(\mathrm{ham}(X), \mathbb{R})^{\mathrm{sp}(X)} \to \Lambda^3 H^1_{\mathrm{dR}}(X)^*$$

is given by

$$T([\psi_\alpha])(b_1, b_2, b_3) = ([\alpha], b_1, b_2, b_3) \tag{91}$$

for the 4-linear map on $H^1_{\mathrm{dR,c}}(X) \times H^1_{\mathrm{dR}}(X)^3$ *defined by*

$$(a, b_1, b_2, b_3) = \int_X a \wedge b_1 \wedge b_2 \wedge b_3 \wedge \omega^{n-2}/(n-2)!\,. \tag{92}$$

Proof. We proceed along the lines of Prop. 6.6. By Corollary 6.4, the class $[\psi_\alpha] \in H^2(\mathrm{ham}(X), \mathbb{R})$ is annihilated by $\mathrm{sp}(X)$ if and only if $[\alpha] \in H^1_{\mathrm{dR},c}(X)$ lies in the kernel of the alternating pairing (87). As $\langle \alpha(v)\rangle$ can be expressed in terms of the pairing as $([\alpha], [i_v\omega])$, this is equivalent to the requirement that $\langle \alpha(v)\rangle = 0$ for all $v \in \mathrm{sp}(X)$. By eqn. (86), we thus have $L_v\psi_\alpha = \mathtt{d}(\theta_\alpha(v))$ for all $v \in \mathrm{sp}(X)$, where the 1-cochain $\theta_\alpha : \mathrm{sp}(X) \to \mathrm{ham}(X)'$ is defined by (85). The differential of any 2-cochain $\psi'_\alpha\colon \mathrm{sp}(X) \times \mathrm{sp}(X) \to \mathbb{R}$ that extends $\theta_\alpha\colon \mathrm{sp}(X) \times \mathrm{ham}(X) \to \mathbb{R}$ is

$$\begin{aligned}
(\mathtt{d}\psi'_\alpha)(v_1, v_2, v_3) &= \sum_{\mathrm{cycl}} \psi'_\alpha(v_1, [v_2, v_3]) = \sum_{\mathrm{cycl}} \theta_\alpha(v_1)([v_2, v_3]) \\
&= \sum_{\mathrm{cycl}} \int_X \alpha(v_1)\,(\omega(v_2, v_3) - \omega(v_2, v_3)_x)\,\omega^n/n! \\
&= \sum_{\mathrm{cycl}} \int_X \alpha(v_1)\omega(v_2, v_3)\omega^n/n! \\
&\overset{(27)}{=} \int_X \alpha \wedge i_{v_1}\omega \wedge i_{v_2}\omega \wedge i_{v_3}\omega \wedge \omega^{n-2}/(n-2)!.
\end{aligned}$$

In the third step, we use that the commutator $[v_1, v_2]$ of symplectic vector fields is Hamiltonian with Hamiltonian function $\omega(v_1, v_2)$, while in the fourth step, we use that $\langle \alpha(v) \rangle = \int_X \alpha(v)\omega^n/n!$ is zero by $\mathrm{sp}(X)$-invariance of $[\psi_\alpha]$. The required expression of the transgression map follows. $\square$

If we consider the bilinear map $(\,\cdot\,,\,\cdot\,)$ of eqn. (87) as a linear map

$$B\colon H^1_{\mathrm{dR},c}(X) \to H^1_{\mathrm{dR}}(X)^*\,,$$

and the 4-linear map $(\,\cdot\,,\,\cdot\,,\,\cdot\,,\,\cdot\,)$ as minus the transgression map

$$T\colon H^1_{\mathrm{dR},c}(X) \to \textstyle\bigwedge^3 H^1_{\mathrm{dR}}(X)^*,$$

then $[\psi_\alpha]$ extends to $\mathrm{sp}(X)$ if and only if $[\alpha] \in H^1_{\mathrm{dR},c}(X)$ lies in $\mathrm{Ker}(B) \cap \mathrm{Ker}(T)$. Since the Kostant-Souriau class $[\psi_{KS}]$ can be described as $[\psi_\alpha]$ with $\alpha = dh$ and $\langle h \rangle = 1$, it always extends to a class on $\mathrm{sp}(X)$. Indeed, an extension is given by the cocycle

$$\psi'_{KS}(v, w) = \omega(v, w)_x.$$

This yields the desired explicit description of the second continuous Lie algebra cohomology of $\mathrm{sp}(X)$.

Theorem 6.13. *Let X be a noncompact connected symplectic manifold. Then we have an isomorphism*

$$H^2(\mathrm{sp}(X), \mathbb{R}) \simeq \textstyle\bigwedge^2(H^1_{\mathrm{dR}}(X)^*) \oplus \mathbb{R}[\psi'_{KS}] \oplus (\mathrm{Ker}(B) \cap \mathrm{Ker}(T))\,,$$

where $\mathrm{Ker}(B) \cap \mathrm{Ker}(T) \subseteq H^1_{\mathrm{dR},c}(X)$ *is the set of classes a such that*

$$\begin{aligned} (a, b) &= 0\,, \\ (a, b_1, b_2, b_3) &= 0 \end{aligned}$$

for all b and b_1, b_2, b_3 in $H^1_{\mathrm{dR}}(X)$.

We apply Theorem 6.13 to a few (classes of) examples, where we calculate $\mathrm{Ker}(B)$ and $\mathrm{Ker}(T)$ using standard methods in algebraic topology.

Example 6.14 (Noncompact surfaces)**.** For a 2-dimensional surface Σ, the map $B\colon H^1_{\mathrm{dR},c}(\Sigma) \to H^1_{\mathrm{dR}}(\Sigma)^*$ is an isomorphism by Poincaré duality. If Σ is noncompact, Theorem 6.13 yields $H^2(\mathrm{sp}(\Sigma), \mathbb{R}) \simeq \bigwedge^2(H^1_{\mathrm{dR}}(\Sigma)^*) \oplus \mathbb{R}[\psi'_{KS}]$.

Also for cotangent spaces, the Roger cocycles are trivial in cohomology.

Corollary 6.15. *Let T^*Q be the cotangent space of a connected manifold Q. Then all Roger cocycles are trivial, and there is an isomorphism*

$$H^2(\mathrm{sp}(T^*Q), \mathbb{R}) \simeq \textstyle\bigwedge^2(H^1_{\mathrm{dR}}(Q)^*) \oplus \mathbb{R}[\psi'_{KS}]\,.$$

Proof. By Poincaré duality, $H^1_{\mathrm{dR},c}(T^*Q)$ is isomorphic to $H^{2n-1}_{\mathrm{dR}}(T^*Q)^*$. Since Q is a deformation retract of T^*Q, we have $H^{2n-1}_{\mathrm{dR}}(T^*Q) \simeq H^{2n-1}_{\mathrm{dR}}(Q)$, so $H^1_{\mathrm{dR},c}(T^*Q) \simeq H^{2n-1}_{\mathrm{dR}}(T^*Q)^* \simeq H^{2n-1}_{\mathrm{dR}}(Q)^*$ vanishes for $n > 1$. For $n = 1$, $H^1_{\mathrm{dR},c}(T^*Q)$ need not vanish, but $\mathrm{Ker}(B) = \{0\}$ by Example 6.14. Since $H^1_{\mathrm{dR}}(Q)^* \simeq H^1_{\mathrm{dR}}(T^*Q)^*$, this concludes the proof. $\square$

Remark 6.16 (Punctured symplectic manifolds)**.** Suppose that $X = M - \{m\}$ is obtained from a *compact* symplectic manifold M by removing a point $m \in M$. Then the Mayer-Vietoris sequence for de Rham cohomology shows that the pullback $\iota^* \colon H^1_{\mathrm{dR}}(M) \to H^1_{\mathrm{dR}}(X)$ by the inclusion $\iota \colon X \hookrightarrow M$ is an isomorphism. Similarly, the pushforward $\iota_* \colon H^1_{\mathrm{dR},c}(X) \to H^1_{\mathrm{dR}}(M)$ is an isomorphism by the Mayer-Vietoris sequence for compactly supported de Rham cohomology. Since every class $[\alpha] \in H^1_{\mathrm{dR}}(X)$ can be represented by a compactly supported 1-form α, the maps $H^1_{\mathrm{dR},c}(X) \times H^1_{\mathrm{dR}}(X) \to \mathbb{R}$ and $H^1_{\mathrm{dR},c}(X) \times \bigwedge^3 H^1_{\mathrm{dR}}(X) \to \mathbb{R}$ induced by equation (87) and (92) agree with the corresponding alternating forms $\bigwedge^2 H^1_{\mathrm{dR}}(M) \to \mathbb{R}$ and $\bigwedge^4 H^1_{\mathrm{dR}}(M) \to \mathbb{R}$ for the compact manifold M.

The following situation affords an example where some, but not all, Roger cocycles extend to the Lie algebra of symplectic vector fields.

Example 6.17 (Thurston's manifold with puncture)**.** Consider the noncompact symplectic manifold $X^4_m := X^4 - \{m\}$, where X^4 is Thurston's symplectic manifold of Example 6.11. By Remark 6.16, we can determine $\mathrm{Ker}(B)$ and $\mathrm{Ker}(T)$ from the cohomology ring $H^\bullet_{\mathrm{dR}}(X^4)$ of the compact manifold, which was calculated in Example 6.11. Recall from equation (90) that

$$H^1_{\mathrm{dR}}(X^4) = \langle [x^*], [p^*], [z^*] \rangle \ .$$

Using the formula $\omega = h^* \wedge x^* + z^* \wedge p^*$ and the fact that Liouville form $\omega^2/2 = z^* \wedge p^* \wedge h^* \wedge x^*$ integrates to 1, we find

$$([x^*], [p^*]) = ([x^*], [z^*]) = 0, \quad ([z^*], [p^*]) = 1 \tag{93}$$

for the alternating pairing (87). It follows that $\mathrm{Ker}(B) = \mathbb{R}[x^*]$ is 1-dimensional, and the $\mathrm{sp}(X^4_m)$-invariant part of $H^2(\mathrm{ham}(X^4_m), \mathbb{R})$ is spanned by a single class $[\psi_{\widetilde{x}^*}]$, with $\widetilde{x}^*$ a compactly supported 1-form on X^4_m cohomologous to x^*. Since the alternating 4-linear map (89) on $H^1_{\mathrm{dR}}(X^4)$ vanishes for dimensional reasons, the invariant class $[\psi_{\widetilde{x}^*}] \in H^2(\mathrm{ham}(X^4_m), \mathbb{R})$ extends to $[\psi'_{\widetilde{x}^*}] \in H^2(\mathrm{sp}(X^4_m), \mathbb{R})$. For the punctured Thurston manifold X^4_m, we thus find

$$H^2(\mathrm{sp}(X^4_m), \mathbb{R}) \simeq \textstyle\bigwedge^2(\mathbb{R}^3) \oplus \mathbb{R}[\psi'_{KS}] \oplus \mathbb{R}[\psi'_{\widetilde{x}^*}] \, .$$

Acknowledgements

C.V. was supported by the grant PN-II-ID-PCE-2011-3-0921 of the Romanian National Authority for Scientific Research. B.J. was supported by the NWO grant 613.001.214 "Generalised Lie algebra sheaves".

This is a pre-copyedited, author-produced version of an article accepted for publication in International Mathematical Research Notices following peer review. The version of record, Int. Math. Res. Not. (2016), Vol. 2016, No. 16, pp. 4996–5047 is available online at: `https://doi.org/10.1093/imrn/rnv301`.

References

[ALDM74] André Avez, André Lichnerowicz, and A. Diaz-Miranda. Sur l'algèbre des automorphismes infinitésimaux d'une variété symplectique. *J. Differential Geometry*, 9:1–40, 1974.

[Bry88] Jean-Luc Brylinski. A differential complex for Poisson manifolds. *J. Differential Geom.*, 28(1):93–114, 1988.

[Bry90] Jean-Luc Brylinski. Noncommutative Ruelle-Sullivan type currents. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 477–498. Birkhäuser Boston, Boston, MA, 1990.

[Die78] Jean Alexandre Dieudonné. *Éléments d'analyse. Tome VII. Chapitre XXIII. Première partie.* Gauthier-Villars, Paris, 1978. Cahiers Scientifiques, Fasc. XL.

[DS49] Jean Dieudonné and Laurent Schwartz. La dualité dans les espaces $(\mathcal{F})$ et $(\mathcal{LF})$. *Ann. Inst. Fourier Grenoble*, 1:61–101 (1950), 1949.

[DWL83a] M. De Wilde and P. B. A. Lecomte. Cohomologie 3-différentiable de l'algèbre de Poisson d'une variété symplectique. *Ann. Inst. Fourier (Grenoble)*, 33(4):83–94, 1983.

[DWL83b] M. De Wilde and P. B. A. Lecomte. Cohomology of the Lie algebra of smooth vector fields of a manifold, associated to the Lie derivative of smooth forms. *J. Math. Pures Appl. (9)*, 62(2):197–214, 1983.

[DWLG84] M. De Wilde, P. Lecomte, and S. Gutt. À propos des deuxième et troisième espaces de cohomologie de l'algèbre de Lie de Poisson d'une variété symplectique. *Ann. Inst. H. Poincaré Sect. A (N.S.)*, 40(1):77–83, 1984.

[Fuk86] D. B. Fuks. *Cohomology of Infinite Dimensional Lie Algebras.* Plenum, New York and London, 1986.

[GKF72] I. M. Gel′fand, D. I. Kalinin, and D. B. Fuks. The cohomology of the Lie algebra of Hamiltonian formal vector fields. *Funkcional. Anal. i Priložen.*, 6(3):25–29, 1972.

[GM92] I. M. Gel′fand and O. Mathieu. On the cohomology of the Lie algebra of Hamiltonian vector fields. *J. Funct. Anal.*, 108(2):347–360, 1992.

[HW41] Witold Hurewicz and Henry Wallman. *Dimension Theory.* Princeton Mathematical Series, v. 4. Princeton University Press, Princeton, N. J., 1941.

[ILM06] Rais S. Ismagilov, Mark Losik, and Peter W. Michor. A 2-cocycle on a symplectomorphism group. *Mosc. Math. J.*, 6(2):307–315, 407, 2006.

[JN15] Bas Janssens and Karl-Hermann Neeb. Projective unitary representations of infinite dimensional lie groups. arxiv:1501.00939, 2015.

[Kir90] A. A. Kirillov. La géométrie des moments pour les groupes de difféomorphismes. In *Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)*, volume 92 of *Progr. Math.*, pages 73–83. Birkhäuser Boston, Boston, MA, 1990.

[Kos70] Bertram Kostant. Quantization and unitary representations. I. Prequantization. In *Lectures in modern analysis and applications, III*, pages 87–208. Lecture Notes in Math., Vol. 170. Springer, Berlin, 1970.

[Lic73] André Lichnerowicz. Cohomologie 1-différentiable d'algèbres de Lie associées à une variété symplectique. *C. R. Acad. Sci. Paris Sér. A-B*, 277:A215–A219, 1973.

[Lic74] André Lichnerowicz. Cohomologie 1-différentiable des algèbres de Lie attachées à une variété symplectique ou de contact. *J. Math. Pures Appl. (9)*, 53:459–483, 1974.

[Mil84] John Willard Milnor. Remarks on infinite-dimensional Lie groups. In *Relativity, groups and topology, II (Les Houches, 1983)*, pages 1007–1057. North-Holland, Amsterdam, 1984.

[MS98] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1998.

[Nee02a] Karl-Hermann Neeb. Central extensions of infinite-dimensional Lie groups. *Ann. Inst. Fourier (Grenoble)*, 52(5):1365–1442, 2002.

[Nee02b] Karl-Hermann Neeb. Universal central extensions of Lie groups. *Acta Appl. Math.*, 73(1-2):175–219, 2002. The 2000 Twente Conference on Lie Groups (Enschede).

[Nee06] Karl-Hermann Neeb. Towards a Lie theory of locally convex groups. *Jpn. J. Math.*, 1(2):291–468, 2006.

[Nom54] Katsumi Nomizu. On the cohomology of compact homogeneous spaces of nilpotent Lie groups. *Ann. of Math. (2)*, 59:531–538, 1954.

[Pal72] Viktor Pavlovich Palamodov. On a Stein manifold the Dolbeault complex splits in positive dimensions. *Mat. Sb. (N.S.)*, 88(130):287–315, 1972.

[Pee60] Jaak Peetre. Réctification (*sic*) à l'article «une caractérisation abstraite des opérateurs différentiels». *Math. Scand.*, 8:116–120, 1960.

[PS86] Andrew Pressley and Graeme Segal. *Loop groups.* Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, 1986. Oxford Science Publications.

[Rog95] Claude Roger. Extensions centrales d'algèbres et de groupes de Lie de dimension infinie, algèbre de Virasoro et généralisations. *Rep. Math. Phys.*, 35(2-3):225–266, 1995. Mathematics as language and art (Białowieża, 1993).

[RS81] Tudor Raţiu and Rudolf Schmid. The differentiable structure of three remarkable diffeomorphism groups. *Math. Z.*, 177(1):81–100, 1981.

[Rud91] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[Sch78] Rudolf Schmid. *Die Symplektomorphismen-Gruppe als Fréchet-Lie-Gruppe*. PhD thesis, Universität Zürich, 1978.

[Sou70] J.-M. Souriau. *Structure des systèmes dynamiques*. Maîtrises de mathématiques. Dunod, Paris, 1970.

[Thu76] W. P. Thurston. Some simple examples of symplectic manifolds. *Proc. Amer. Math. Soc.*, 55(2):467–468, 1976.

[Trè67] François Trèves. *Topological vector spaces, distributions and kernels*. Academic Press, New York-London, 1967.

[vdK73] W. L. J. van der Kallen. *Infinitesimally central extensions of Chevalley groups*. Lecture Notes in Mathematics, Vol. 356. Springer-Verlag, Berlin, 1973.

[Viz06] Cornelia Vizman. Central extensions of the Lie algebra of symplectic vector fields. *J. Lie Theory*, 16(2):297–309, 2006.